Thermo-viscous instability of flow in a weakly heat-conducting channel

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Thermo-viscous instability in literature

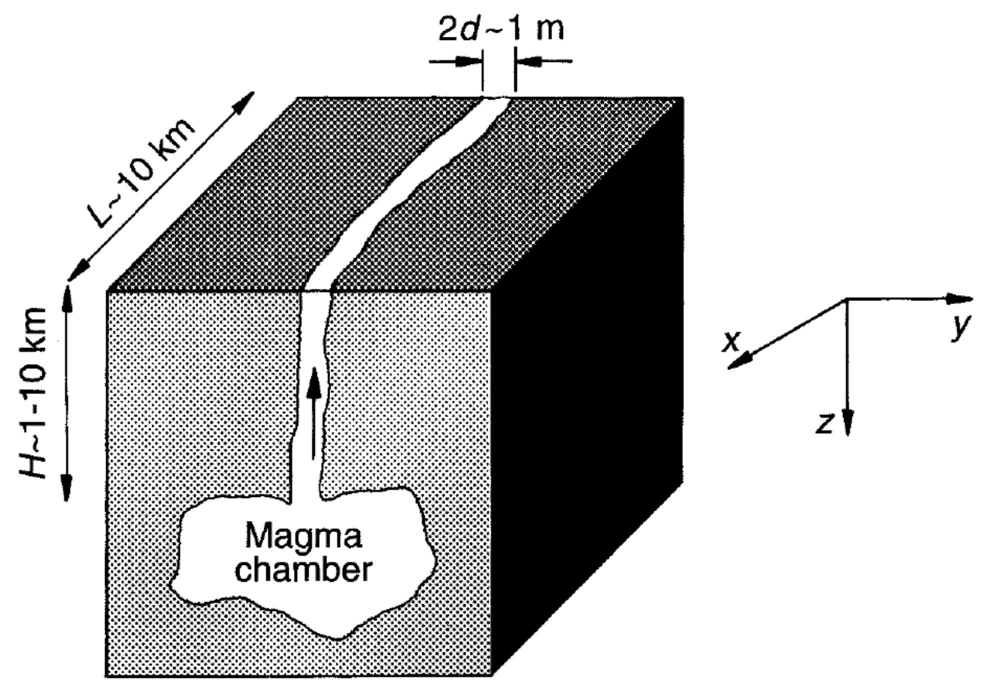
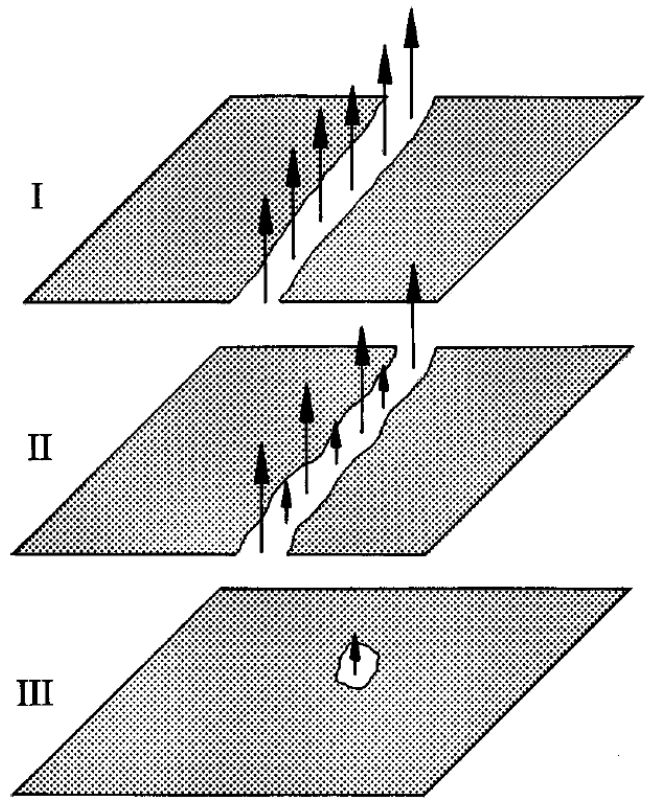
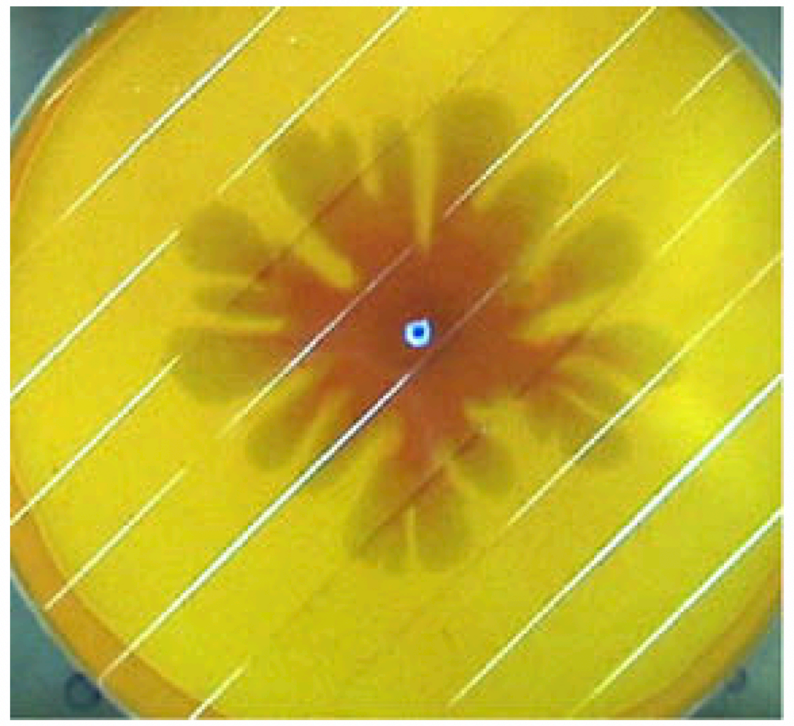


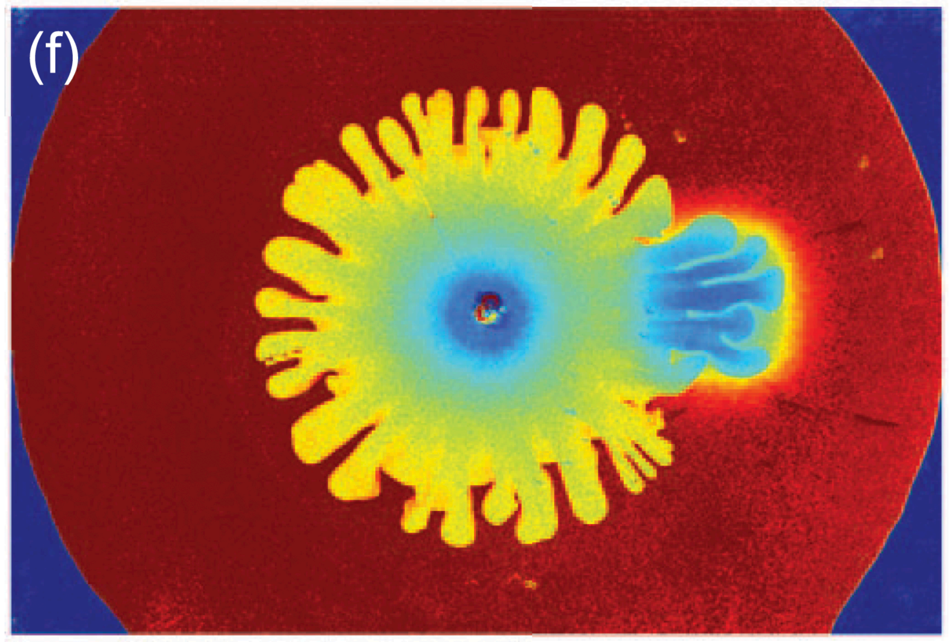
Fig. 1 Cross-section of a planar fissure



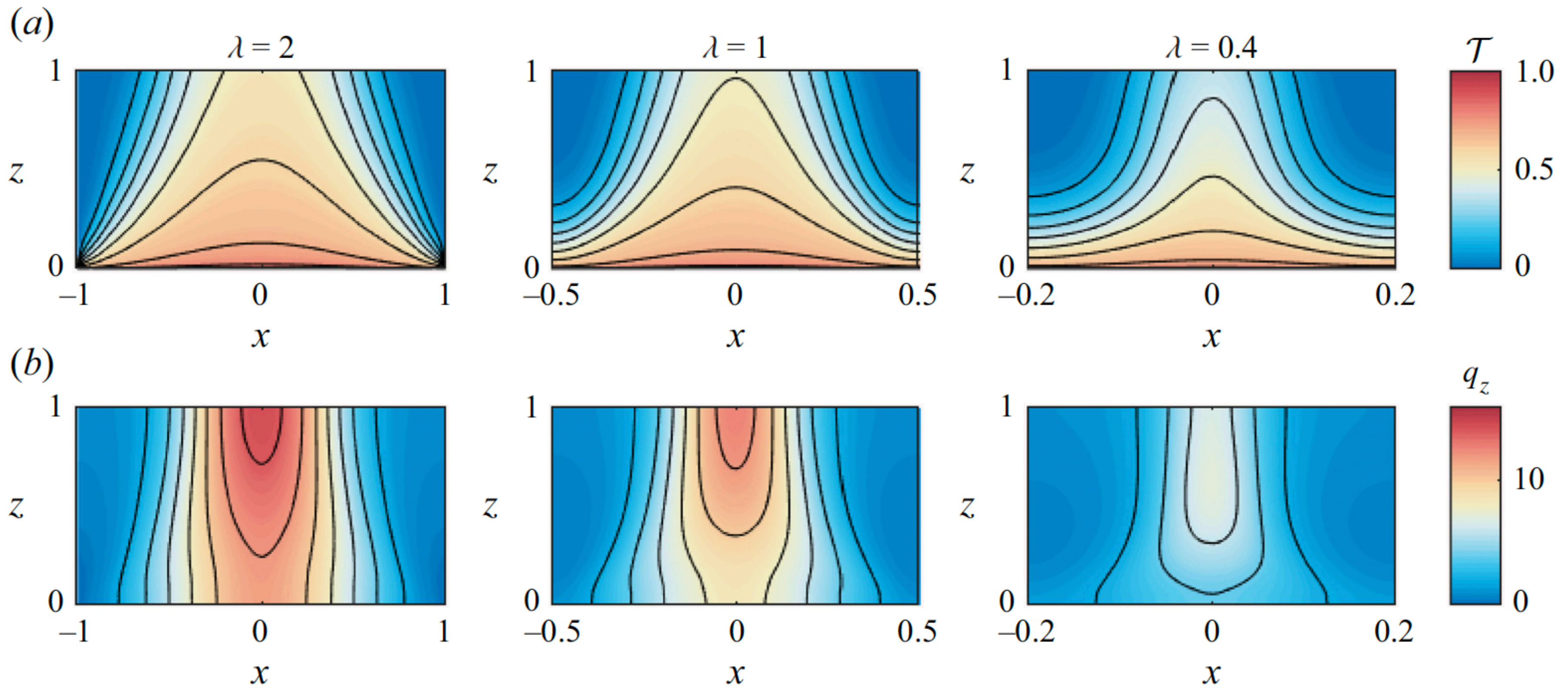
Wylie et al., Bull. Vulcanol. **60** (1999)



Nagatsu et al., Exp. Th. Fl. Sc. **33** (2009)



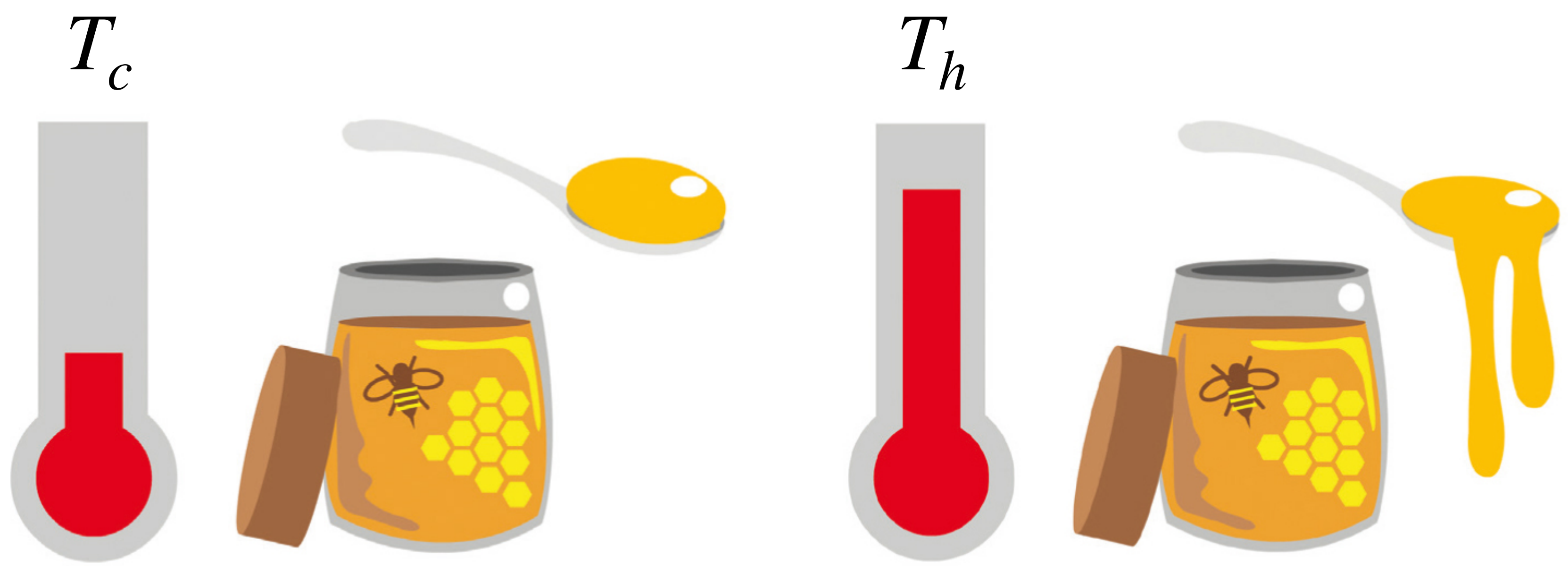
Bunton et al., Phys. Fl. **26** (2014)



Taylor-West et al., J. Fluid Mech. **1015** (2025)

- Hot fluids often focalizes in “fingers” when flowing in a cold, confined structure (ex. magma in a fracture, glycerol in a Hele-shaw cell)
- Saffman-Taylor-like instability, but involving only one fluid
- Key aspects:
  (i) temperature-dependent viscosity
  (ii) confined geometry

Thermo-viscous fingering (intuitively)



First-order Arrhenius law for viscosity $\mu(T)$:

$$\mu(T) = \beta^{T}, \quad \beta = \frac{\mu(T_h)}{\mu(T_c)}$$ (Viscosity ratio)

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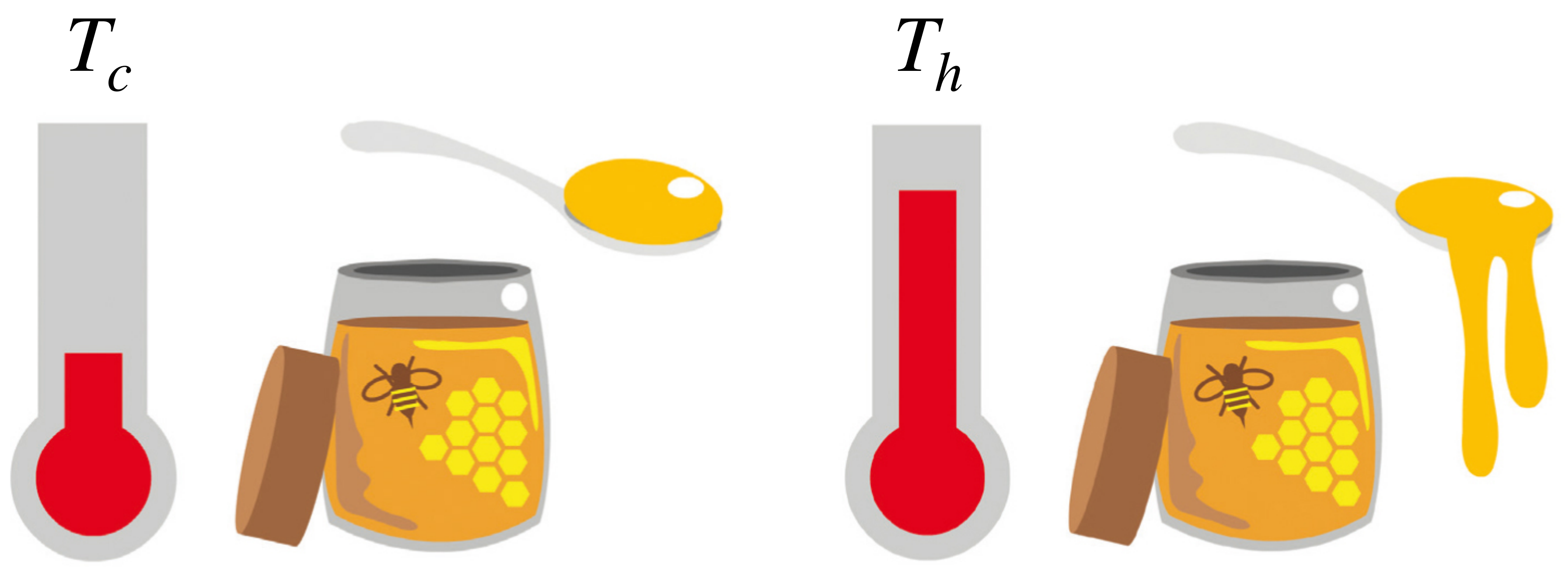

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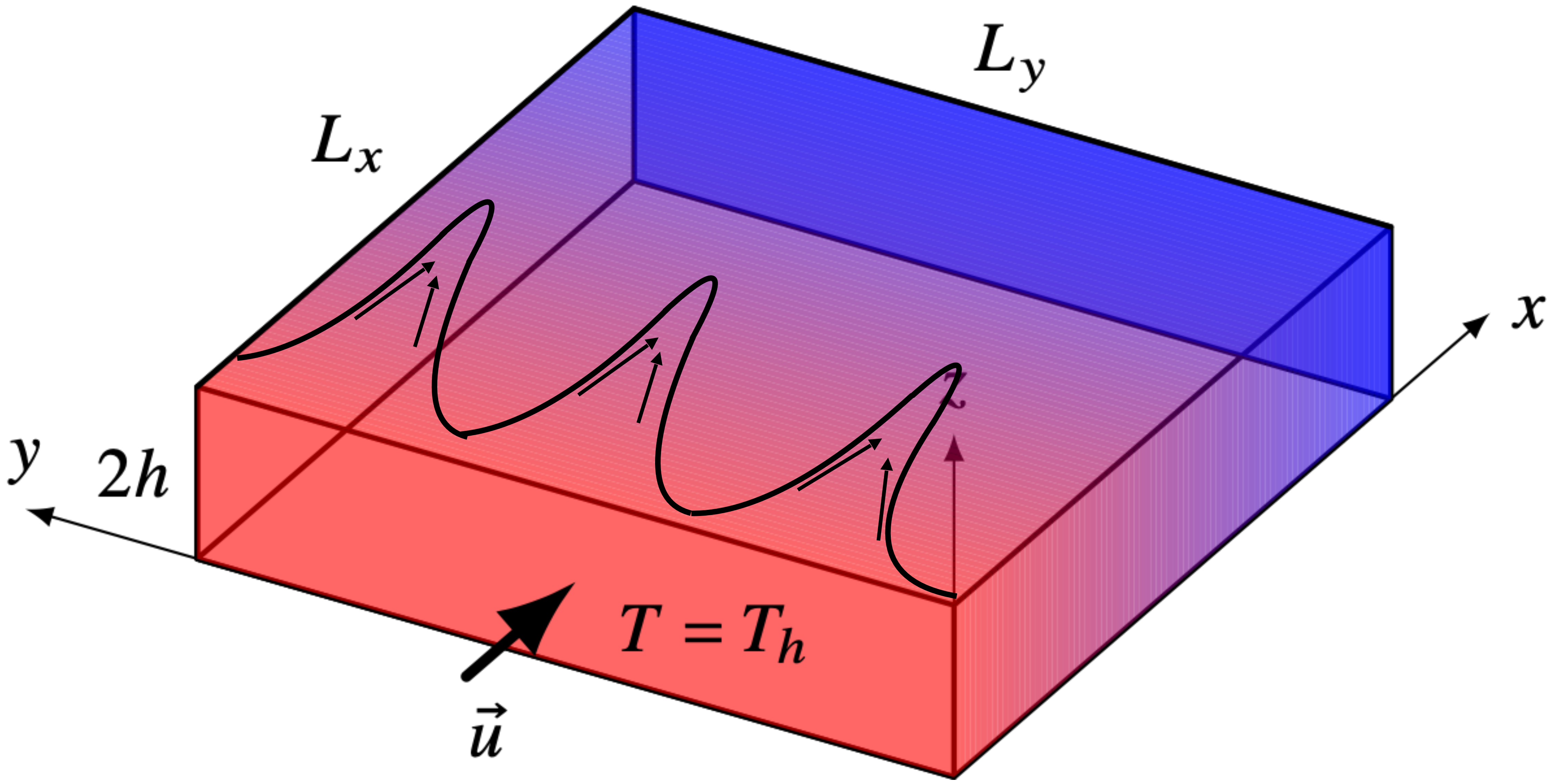
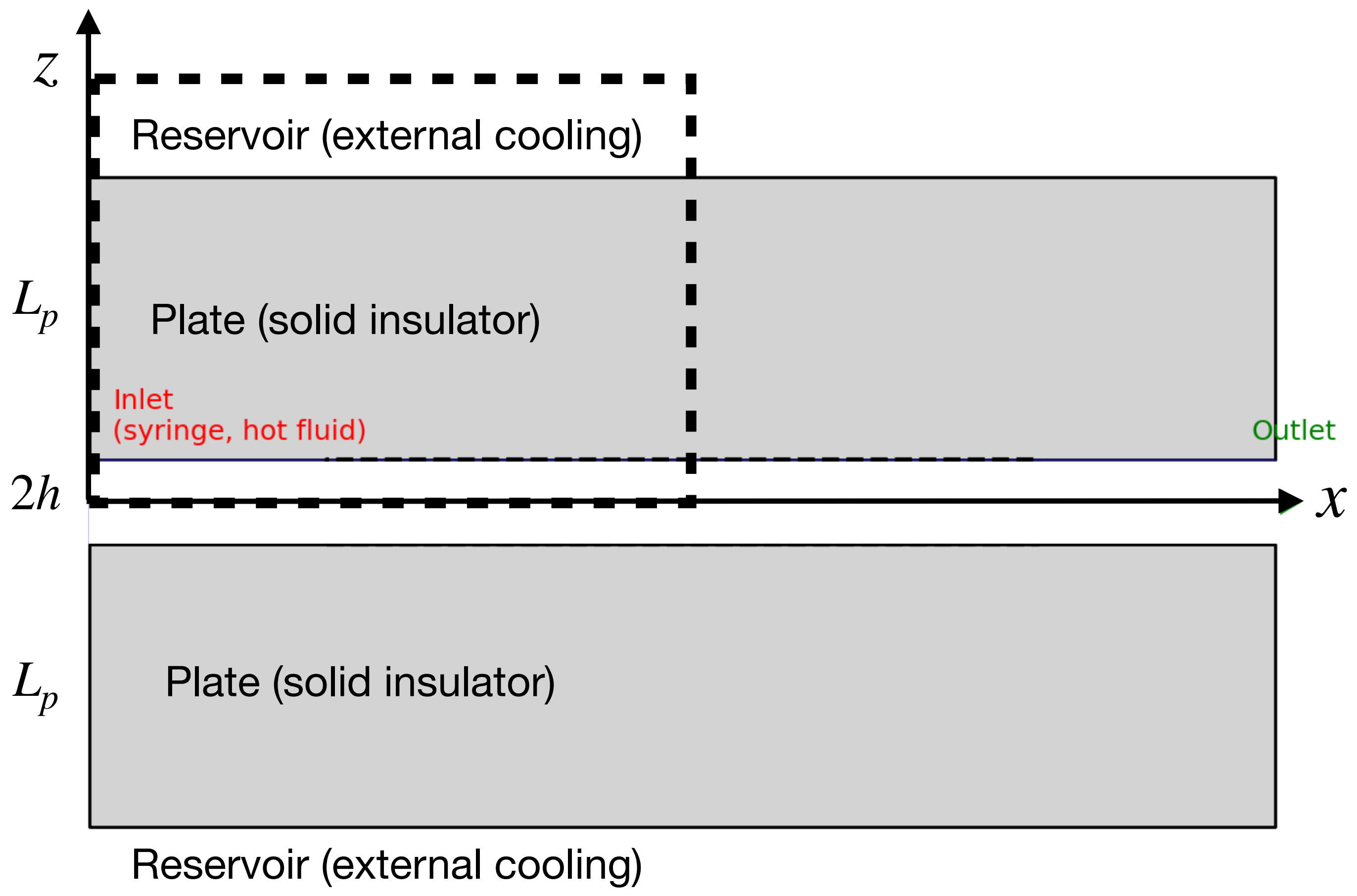


Invasion of hot fluid in a cell with cold fluid:

Slightly higher $\vec{u}$ ⇨ T is transported more efficiently ⇨ $\mu(T) \searrow$ ⇨ $\vec{u}$ remains high

Slightly lower $\vec{u}$ ⇨ T is transported less efficiently ⇨ $\mu(T) \nearrow$ ⇨ $\vec{u}$ shrinks

Novelty: Long-time asymptotic



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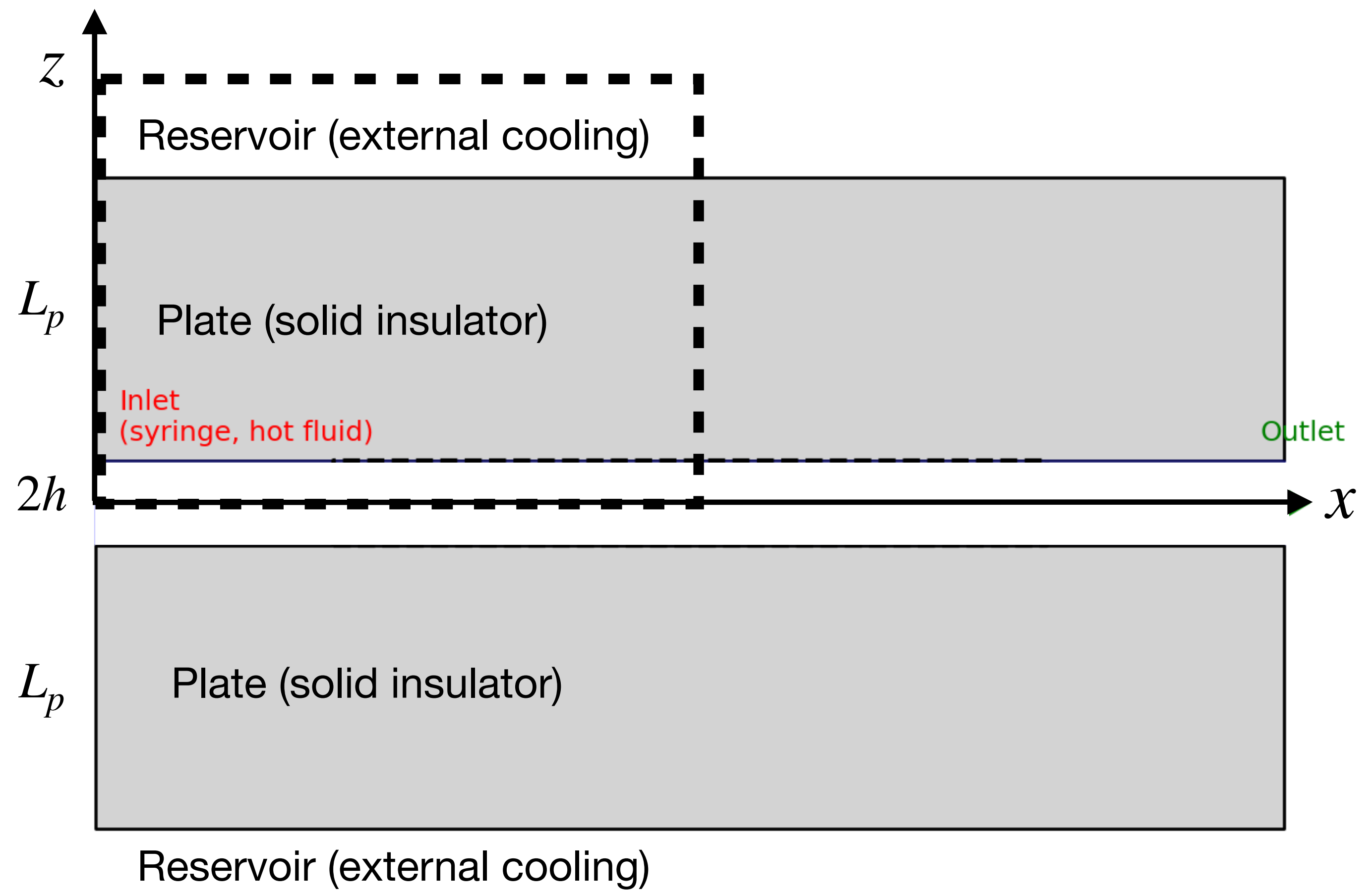


Thermal diffusivity

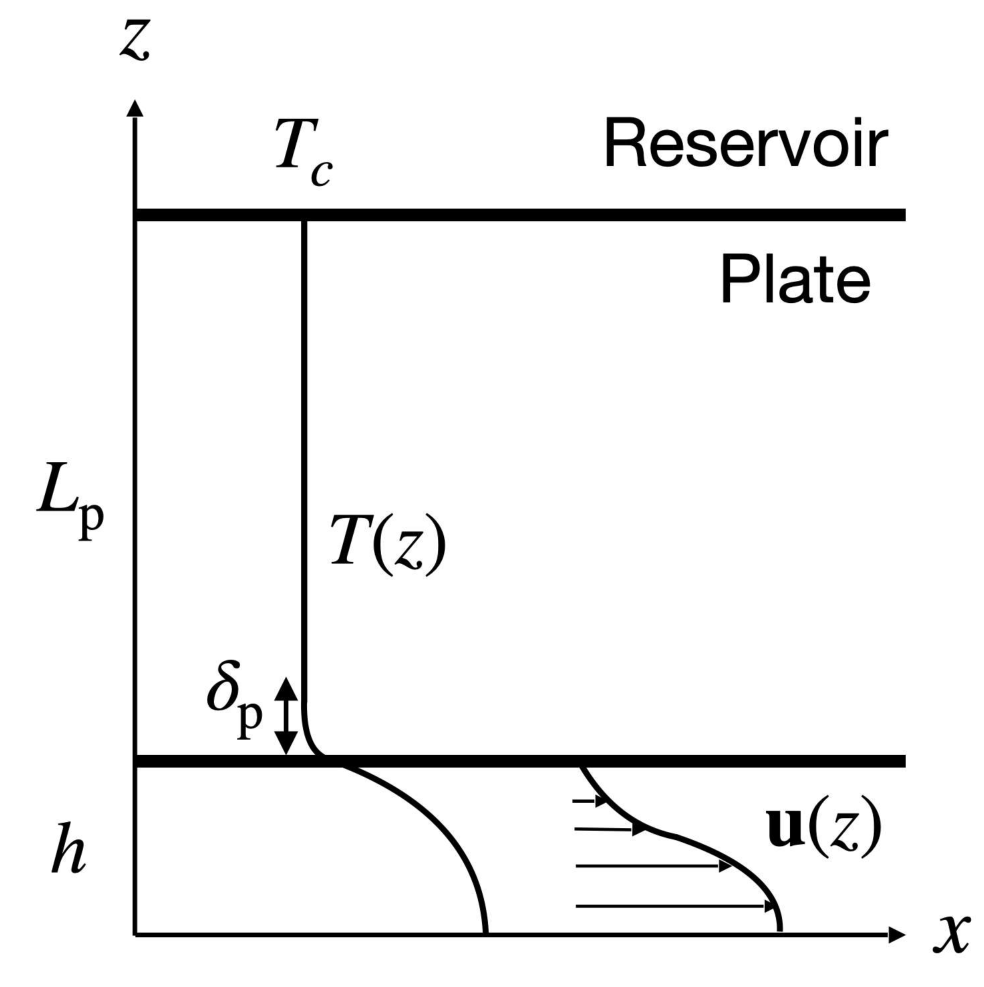
Heat conductivity

$$\kappa_i = \frac{k_i}{\rho_i c_i} \quad (i = f, p)$$

Density

Specific heat capacity

Short time: $t \sim h^2/\kappa_f$



$T(z = h) = T_c$ (Dirichlet b.c.)

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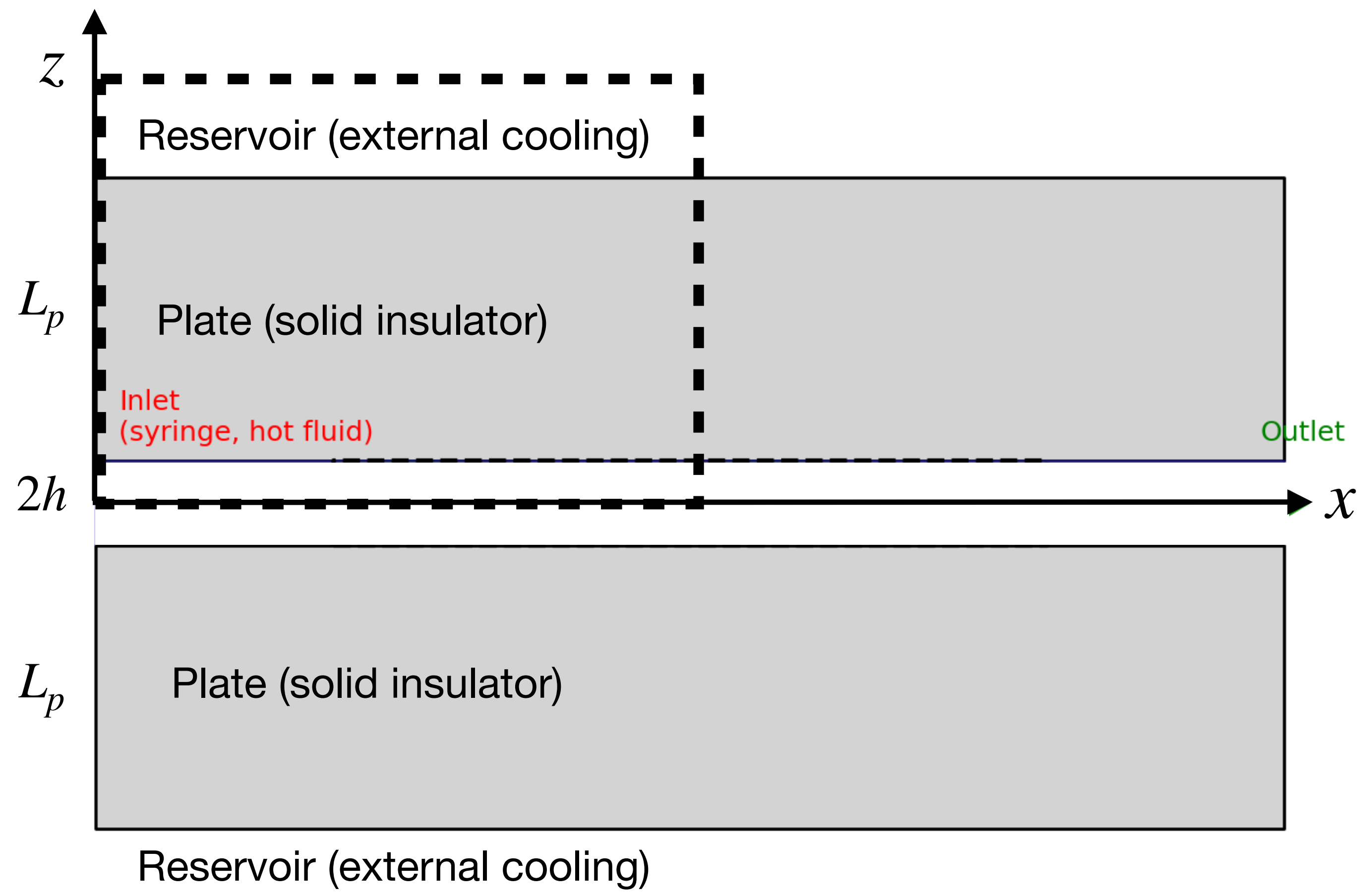

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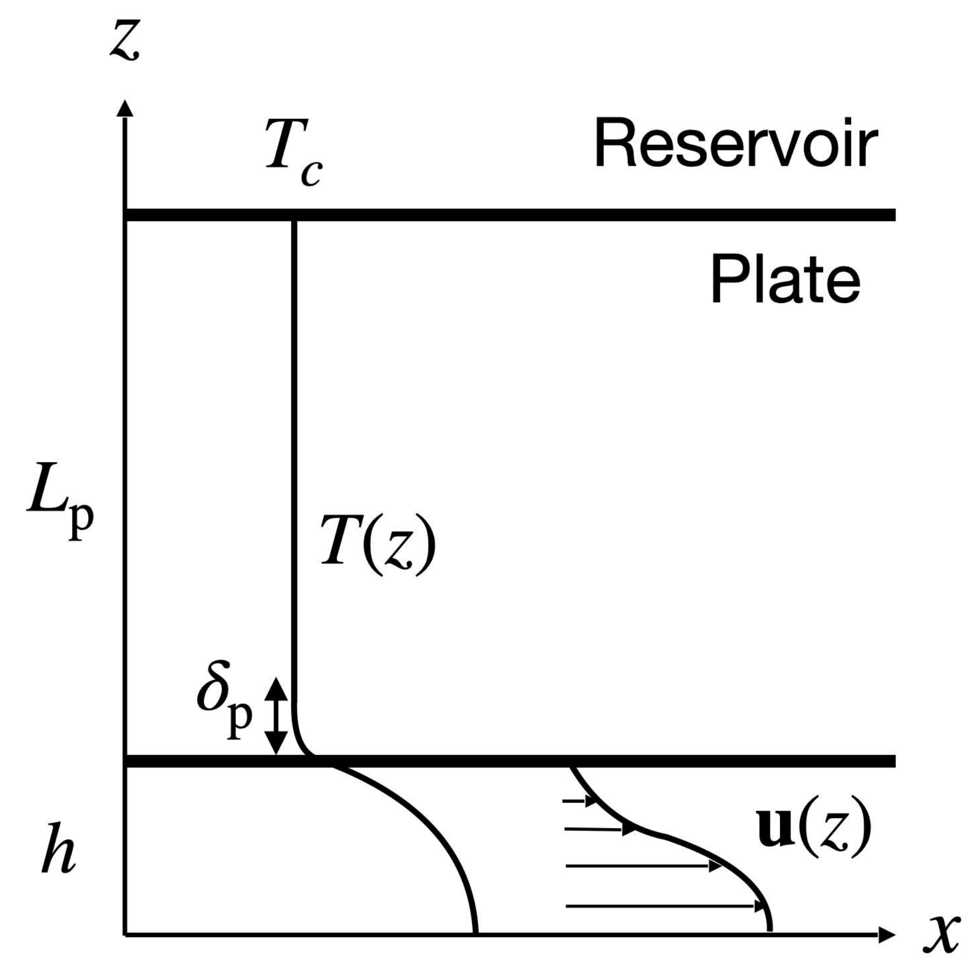
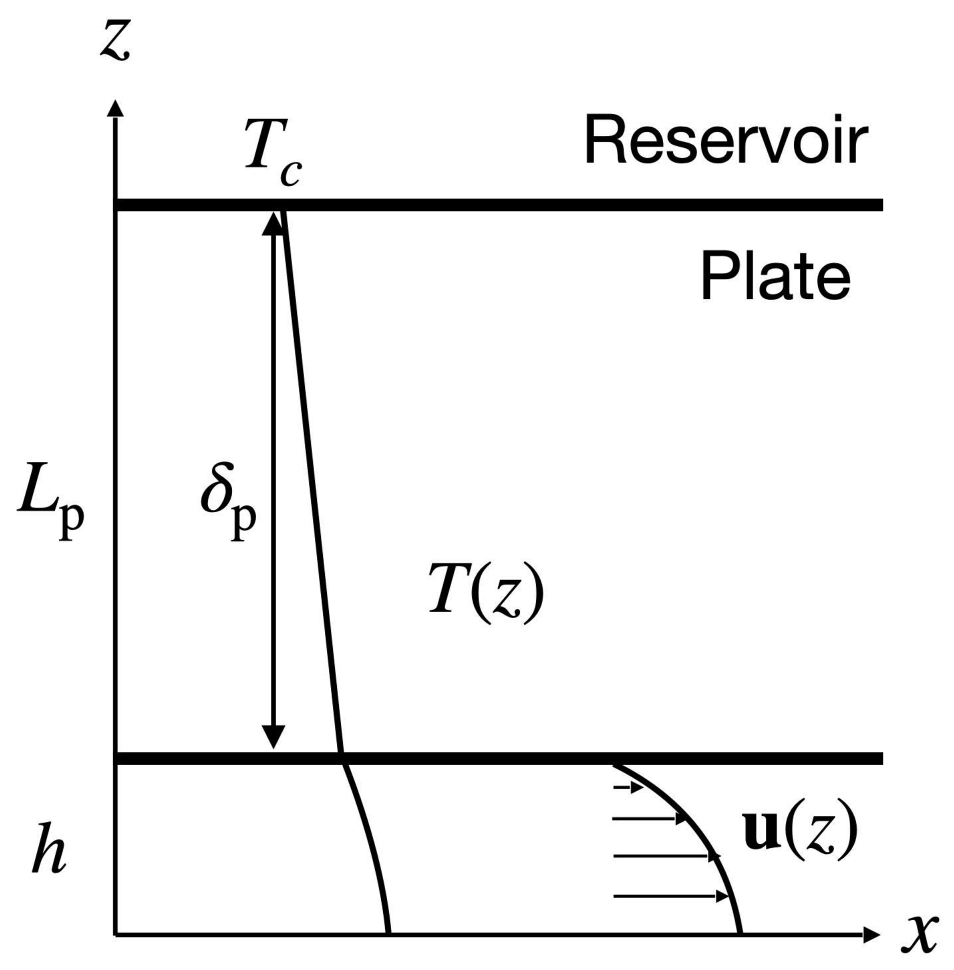


$$T(z = h) = T_c \text{ (Dirichlet b.c.)}$$

Long time: $t \gtrsim L_p^2/\kappa_p \gg h^2/\kappa_f$



$$\left.\frac{\partial T}{\partial z}\right|_{z=h} = \frac{H_\mathrm{ov}}{k_p}(T_c - T) \text{ (Robin b.c.)}$$

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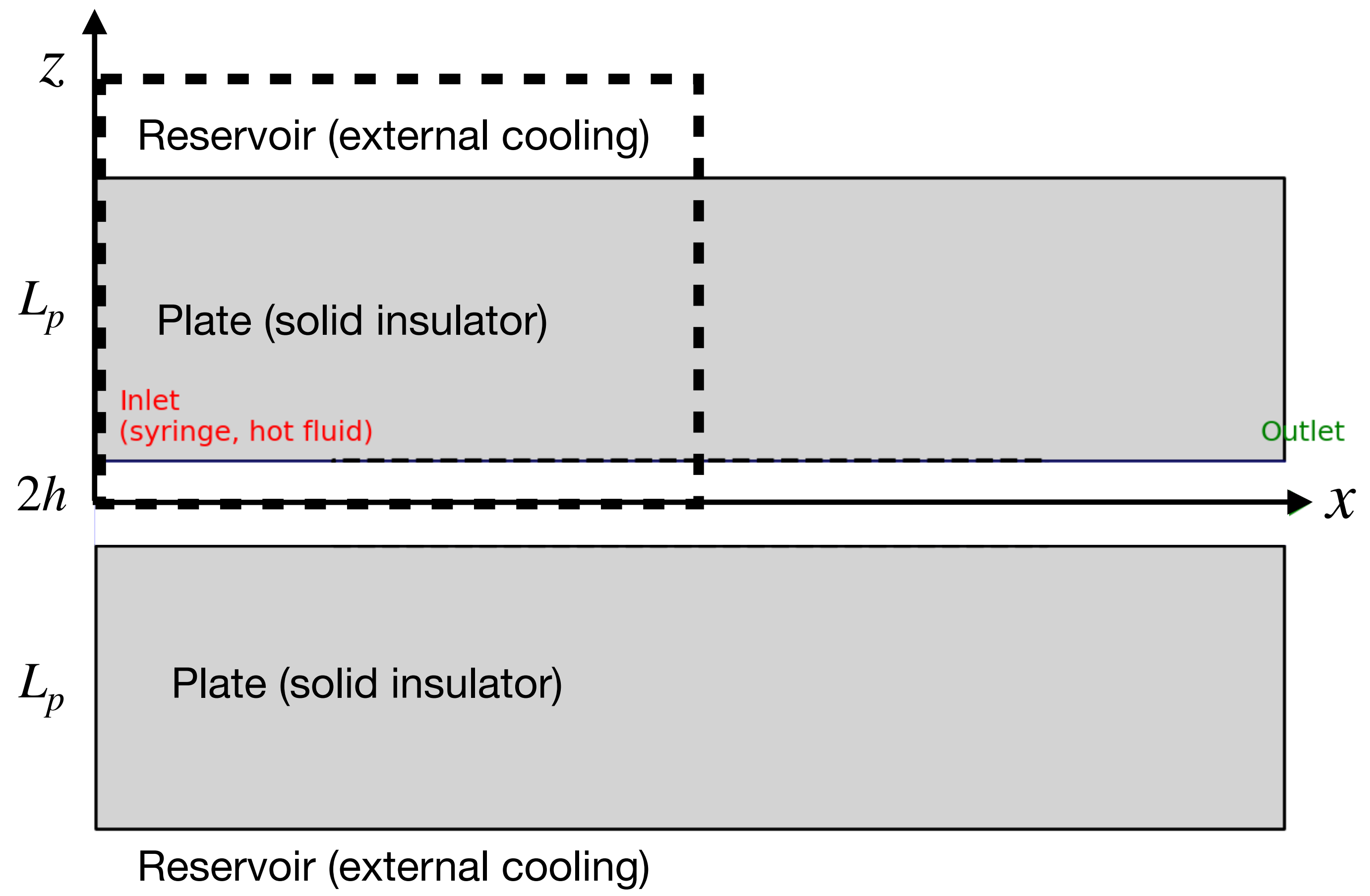

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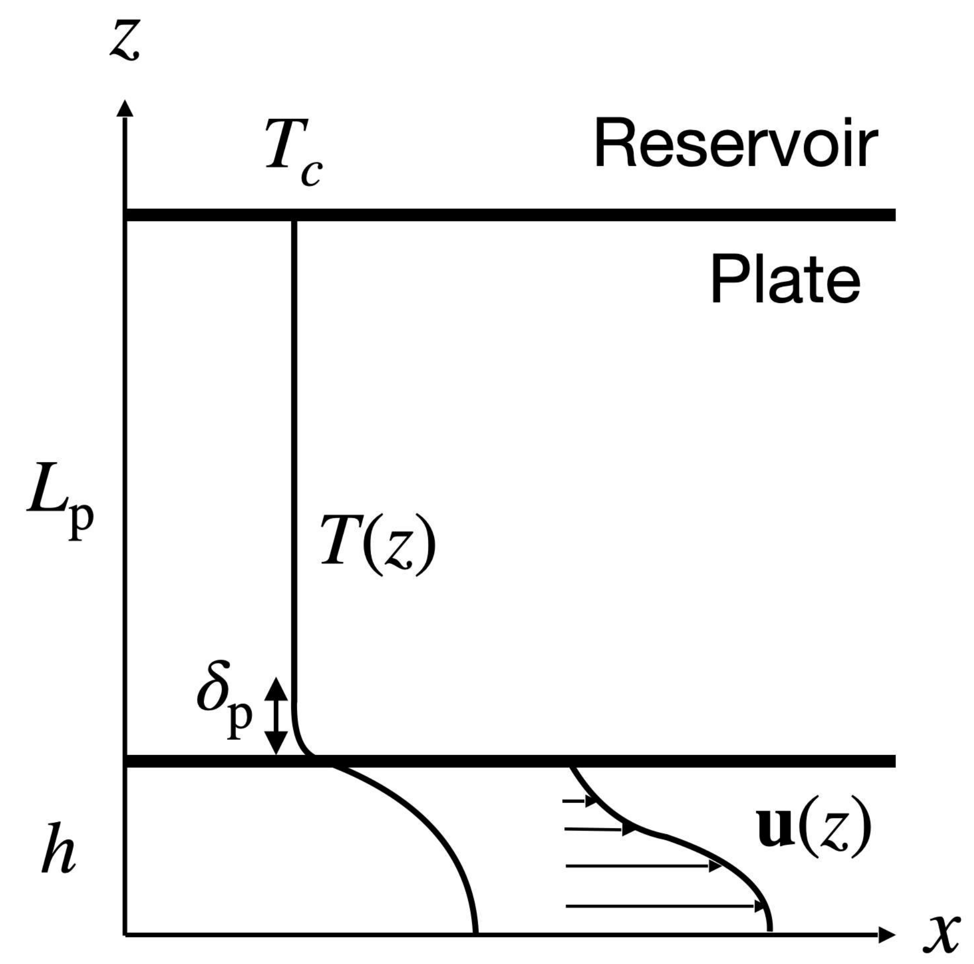

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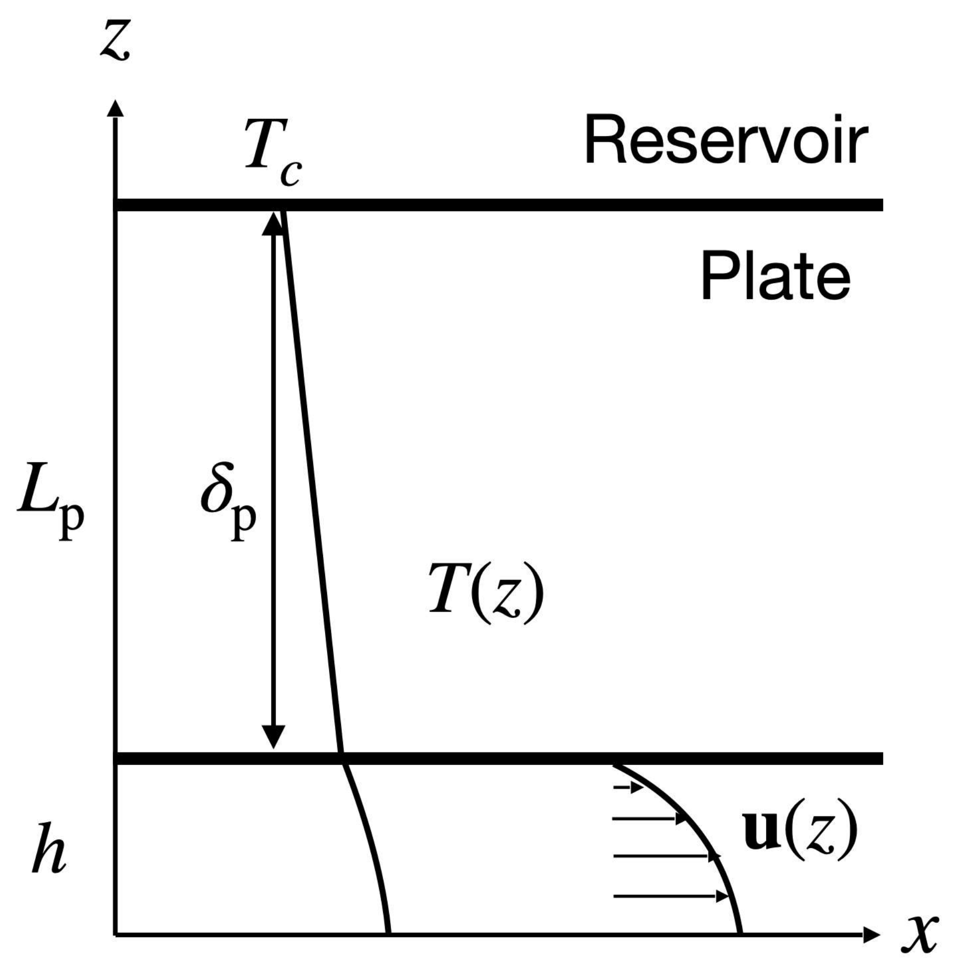

$\left.\frac{\partial T}{\partial z}\right|_{z=h} = \frac{H_\mathrm{ov}}{k_p}(T_c - T)$ (Robin b.c.)

- Heat transfer coefficient:

$$H_\mathrm{ov} = \left(\frac{1}{H_\mathrm{f}} + \frac{\delta_\mathrm{p}}{k_\mathrm{p}} + \frac{1}{H_\mathrm{res}}\right)^{-1} \overset{t \gtrsim \frac{L_p^2}{\kappa_p}}{\approx} \frac{k_\mathrm{p}}{L_\mathrm{p}}$$

- Biot number:

$$\mathrm{Bi} = \frac{H_\mathrm{ov} h}{k_f} \overset{t \gtrsim \frac{L_p^2}{\kappa_p}}{\approx} \frac{h}{L_p}\frac{k_p}{k_f} \ll 1$$

Problem and inlet condition

Equations for cross-gap averaged 2D system (Full problem):

(i) Mass balance (incompressibility)

$$\nabla \cdot \mathbf{u} = 0$$

(ii) Momentum balance (Darcy law)

$$\mathbf{u} = -\beta^{-T}\nabla p$$

(iii) Energy balance (advection-diffusion)

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T - \nabla\left[\left(\mathrm{Pe}^{-1}\,\mathbf{I} + \frac{2\mathrm{Pe}}{105}\,\mathbf{u}\otimes\mathbf{u}\right)\nabla T\right] + \Gamma T = 0$$

BC: - Inlet $(x=0)$**:** $T = T_h$ - Outlet $(x = L_x)$**:** $\nabla T|_{L_x} = 0,\, p = 0$

Global parameters:

- Viscosity ratio: $\beta = \dfrac{\mu(T_h)}{\mu(T_c)}$ - Cooling rate**:** $\Gamma = \dfrac{H_{\mathrm{ov}}}{\rho_f c_f U}$
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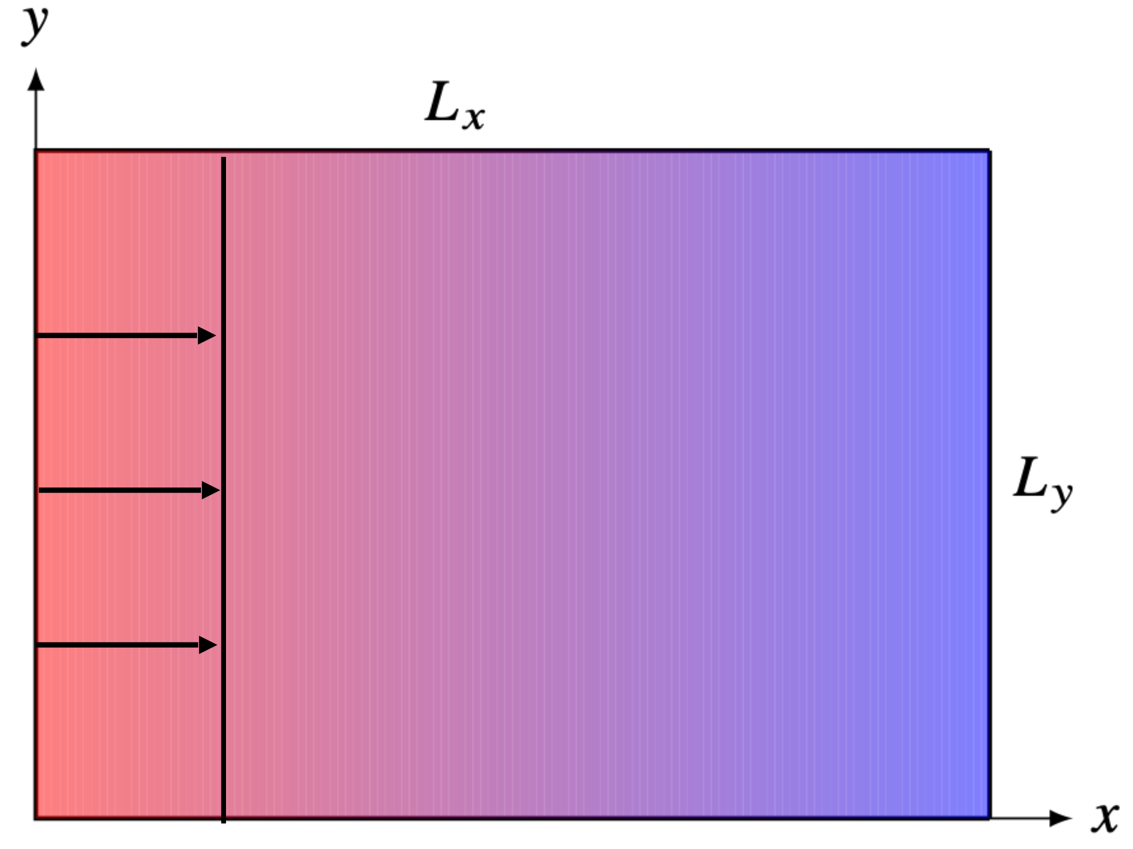
Base state:

$$\mathbf{u}_0 \equiv \hat{\mathbf{e}}_x, \quad p_0(x) = -\int_0^x \beta^{e^{\xi x'}} dx' + C,$$

$$T_0(x) = e^{-\xi x}$$

Thermal entry length $1/\xi$:

$$\xi = \frac{-1 + \sqrt{1 + 4\Gamma(\mathrm{Pe}^{-1} + 2\mathrm{Pe}/105)}}{2(\mathrm{Pe}^{-1} + 2\mathrm{Pe}/105)}$$



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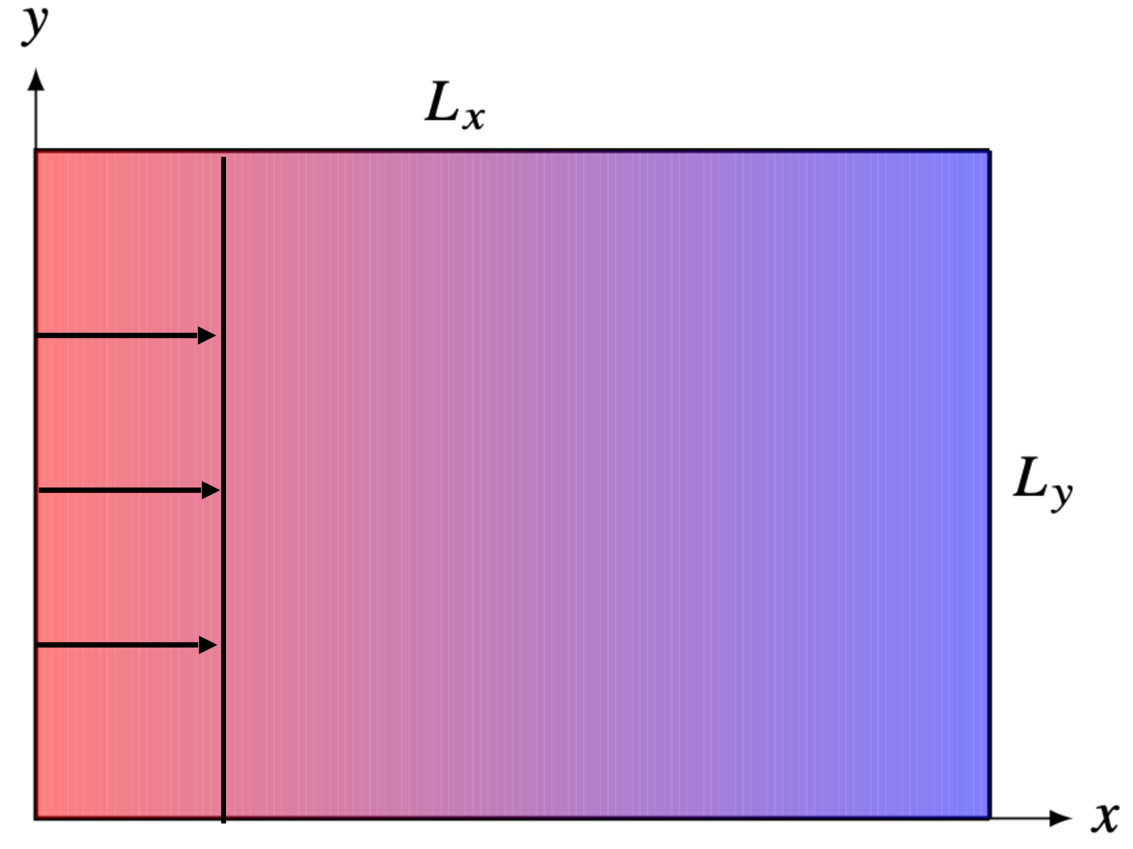
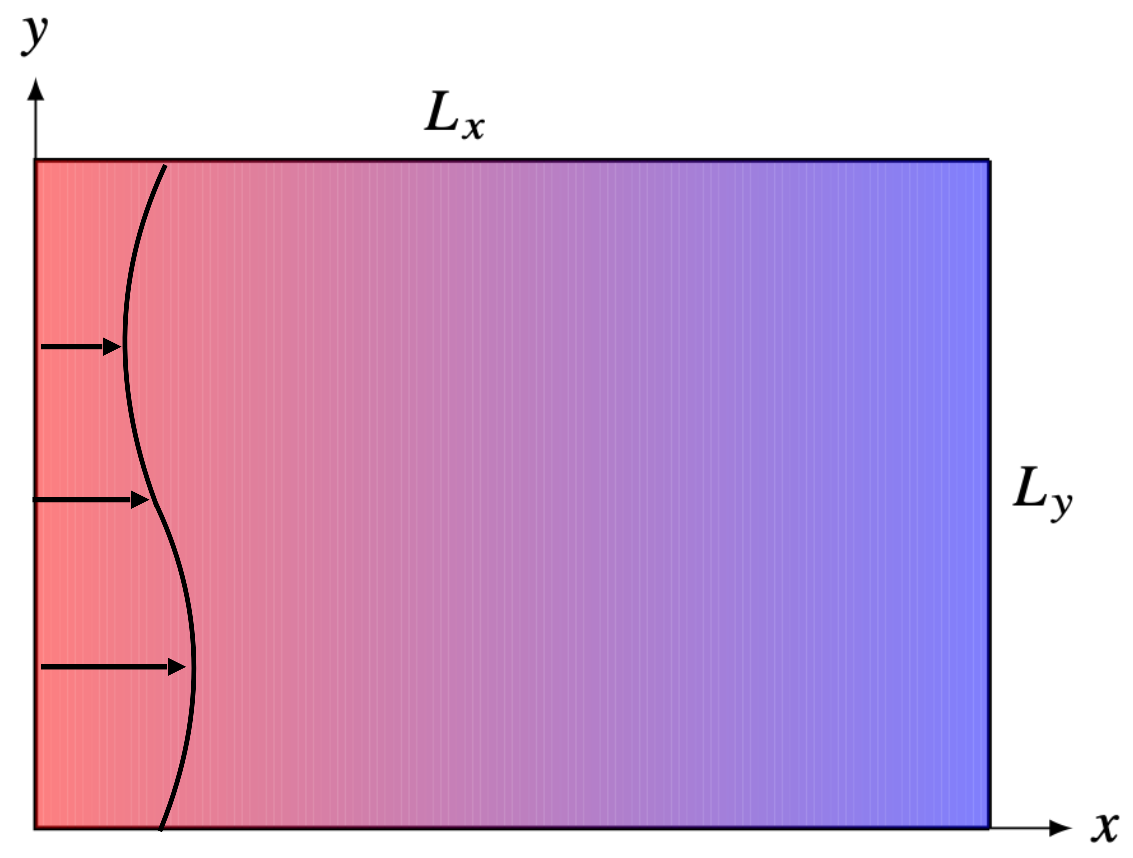


Perturbation:

$$\mathbf{u}(x = 0, y, t) = u_x \hat{\mathbf{e}}_x,$$

$$u_x = \begin{cases} 1 + \epsilon f(x) & \text{for } t < t_{\mathrm{pert}} \\ 1 & \text{for } t > t_{\mathrm{pert}} \end{cases}$$



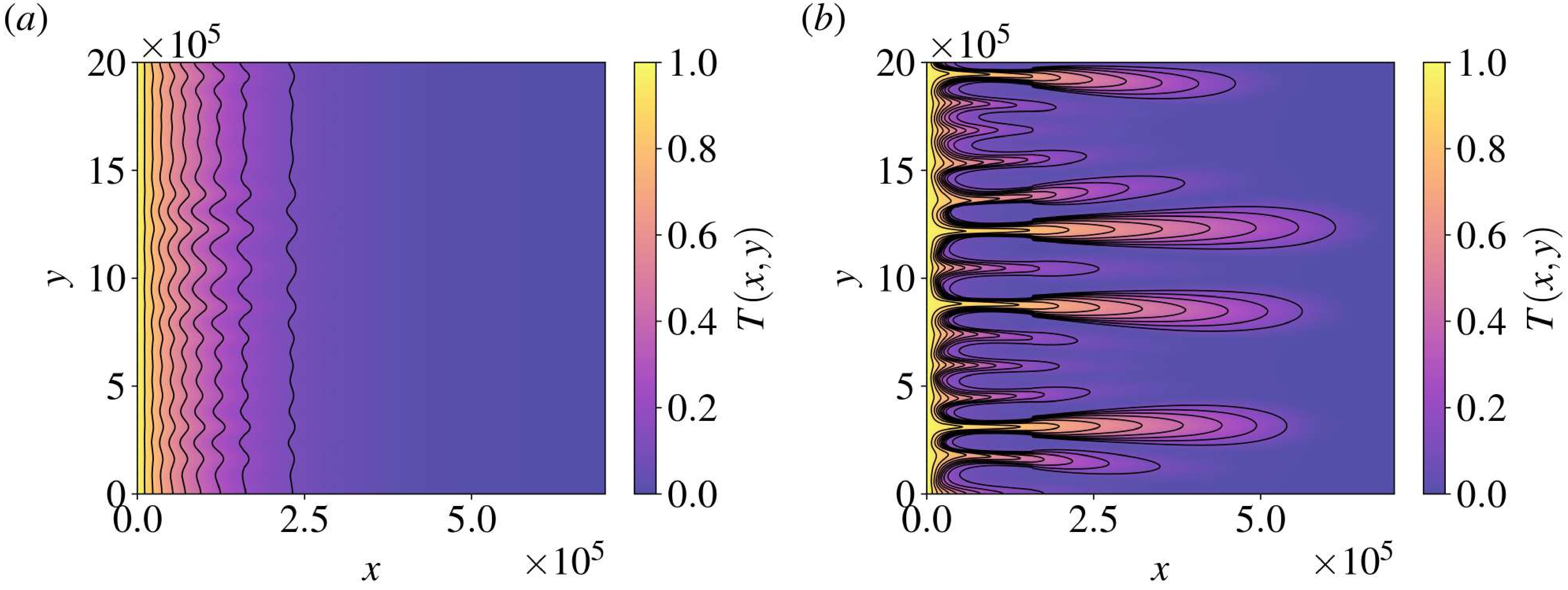
Random perturbation

$$\mathbf{u}(x=0,y,t) = u_x \hat{\mathbf{e}}_x,$$

$$u_x = \begin{cases} 1 + \epsilon\, \eta(y) & \text{for } t \le t_{\text{pert}} \\ 1 & \text{for } t > t_{\text{pert}}, \end{cases}$$

$pdf(\eta)$ Gaussian, $\langle \eta \rangle = 0$, $\langle \eta^2 \rangle = 1$

Snapshots of T at different times



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- Quantify the instability growth:

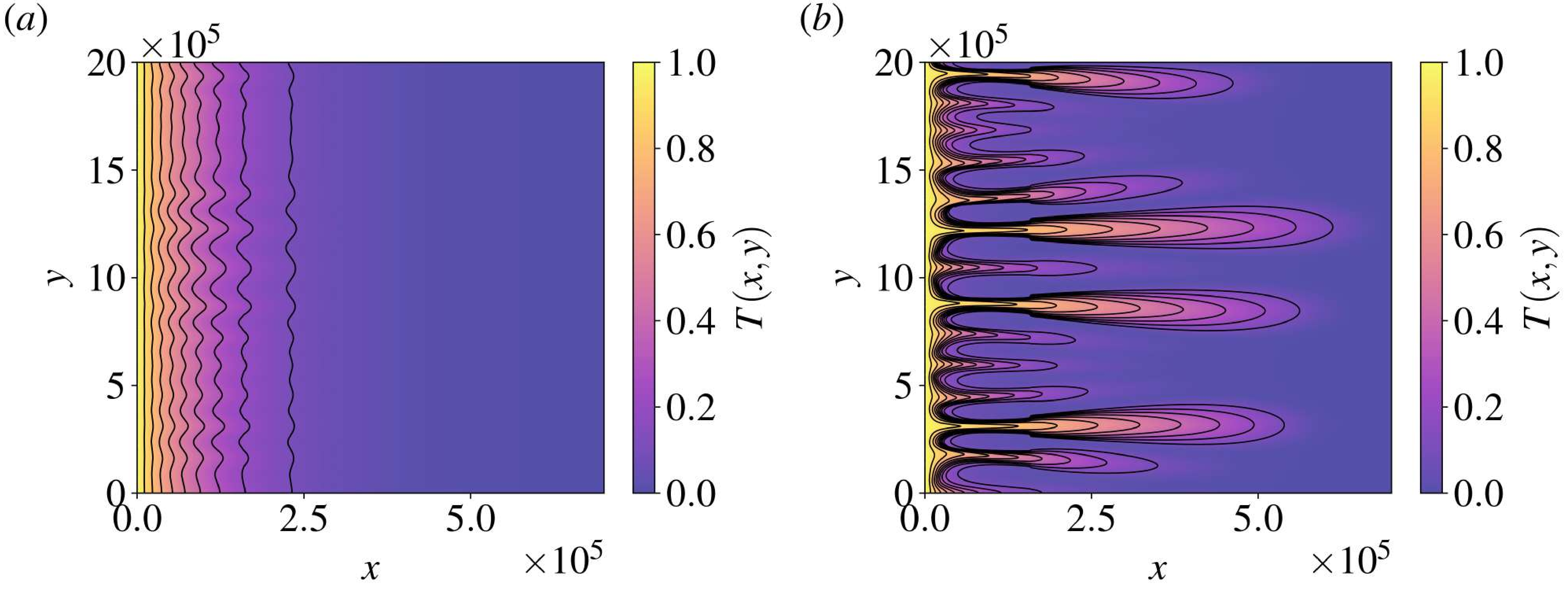
$$T^{\text{span}}(x,t) = T^{\max}(x,t) - T^{\min}(x,t)$$
$$u_x^{\text{span}}(x,t) = u_x^{\max}(x,t) - u_x^{\min}(x,t)$$

- It comes out that:

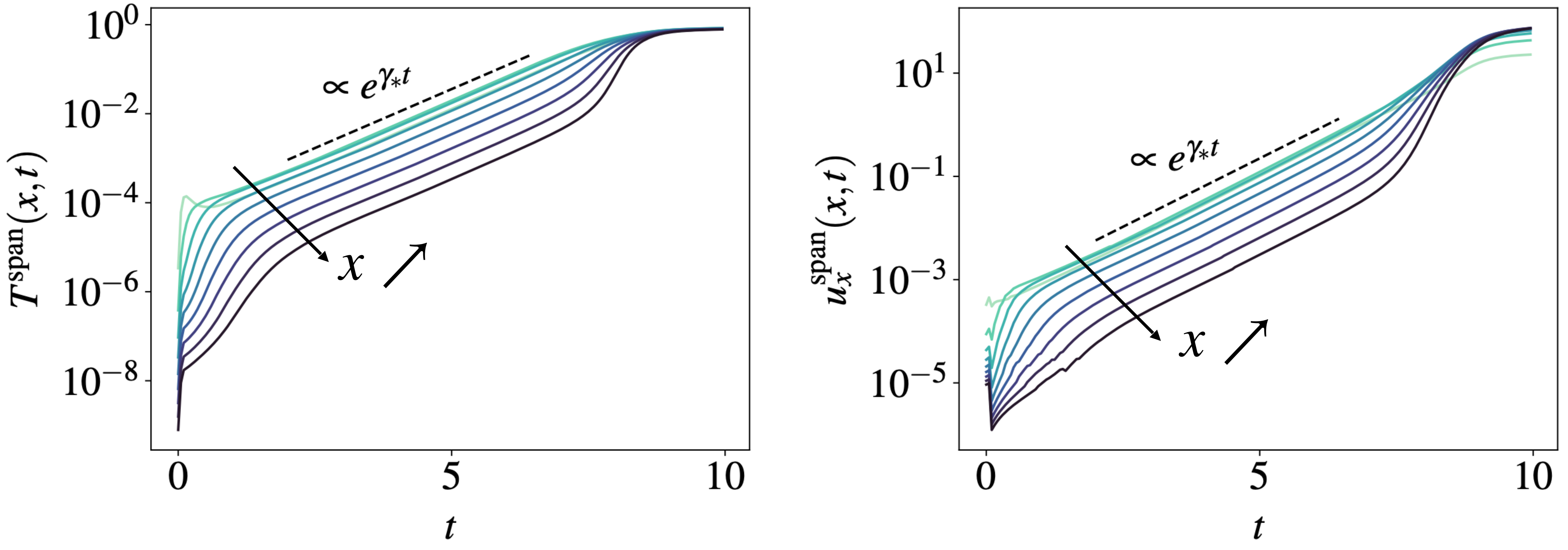
$$T^{\text{span}}, u_x^{\text{span}} \propto \begin{cases} e^{\gamma^* t} & \text{for t} \lesssim \text{t}_{\text{stat}} \\ \text{const for t} \gtrsim \text{t}_{\text{stat}} \end{cases}$$

Growth rate $\gamma^* = \gamma^*(\beta, \Gamma, \text{Pe})$
Wavenumber $k^* = k^*(\beta, \Gamma, \text{Pe})$

Snapshots of T at different times



$T^{\text{span}}, u_x^{\text{span}}$ vs time



Linear Stability Analysis (LSA)

Perturbation around the base state:

$$T(x, y, t) = T_0(x) + T'(x, y, t)$$

$$\mathbf{u}(x, y, t) = \mathbf{u}_0(x) + \mathbf{u}'(x, y, t)$$

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$$T', \ |\mathbf{u}'|, \ p' \propto \epsilon \ll 1$$

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Linearized problem for $T'(x,y,t) = \hat{T}_k(x)e^{iky}e^{\gamma t},\ u'_x(x,y,t) = \hat{u}_k(x)e^{iky}e^{\gamma t}$:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \log\beta\frac{\mathrm{d}T_0}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x} - k^2\right)\hat{u}_k = k^2\log\beta\,\hat{T}_k$$

$$\left(\gamma + \frac{\mathrm{d}}{\mathrm{d}x} - \kappa_{\mathrm{eff}}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{Pe}^{-1}\mathrm{k}^2 + \Gamma\right)\hat{T}_k = \left(-\frac{\mathrm{d}T_0}{\mathrm{d}x} + \frac{2\mathrm{Pe}}{105}\left(2\frac{\mathrm{d}^2T_0}{\mathrm{d}x^2} + \frac{\mathrm{d}T_0}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\right)\right)\hat{u}_k$$

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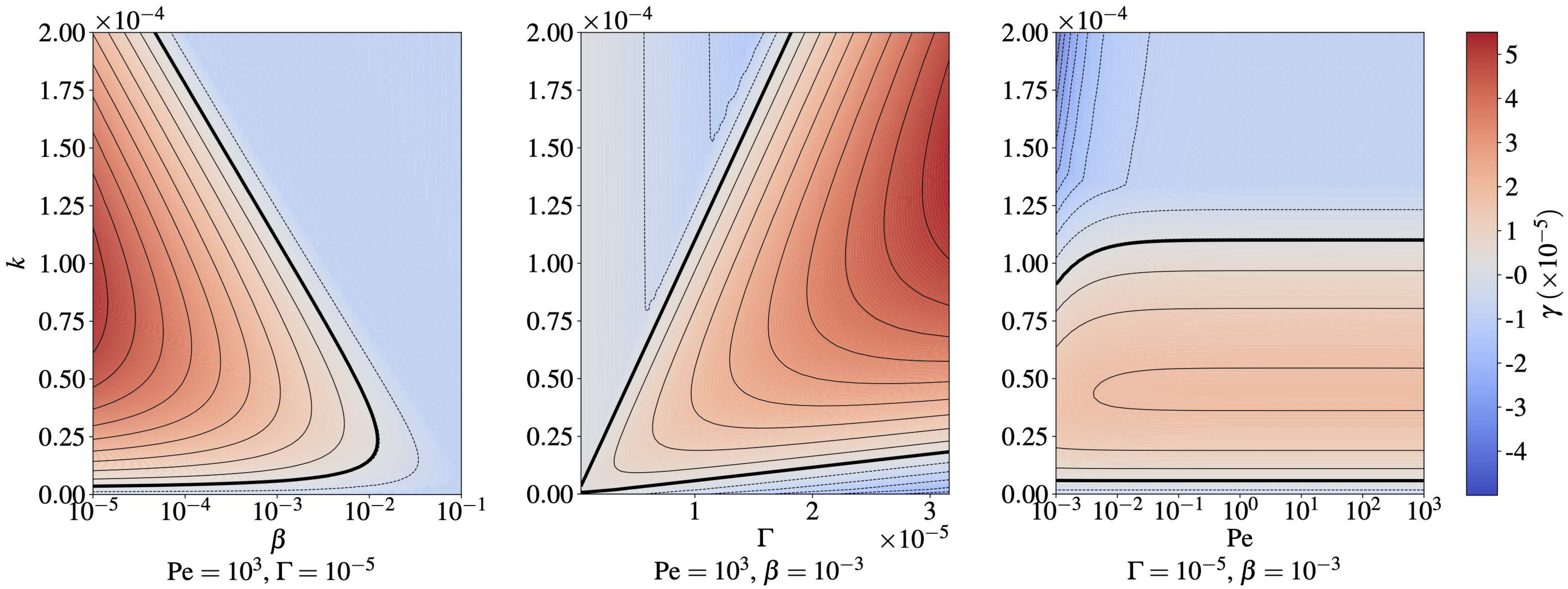
Dispersion relationship:

$\gamma = \gamma(\beta, \Gamma, \mathrm{Pe}, k)$

- Maximum (most dangerous) growth rate and wavelength:

$$\left.\frac{d\gamma}{dk}\right|_{k=k_{\max}} = 0, \quad \gamma_{\max} = \gamma(k_{\max})$$

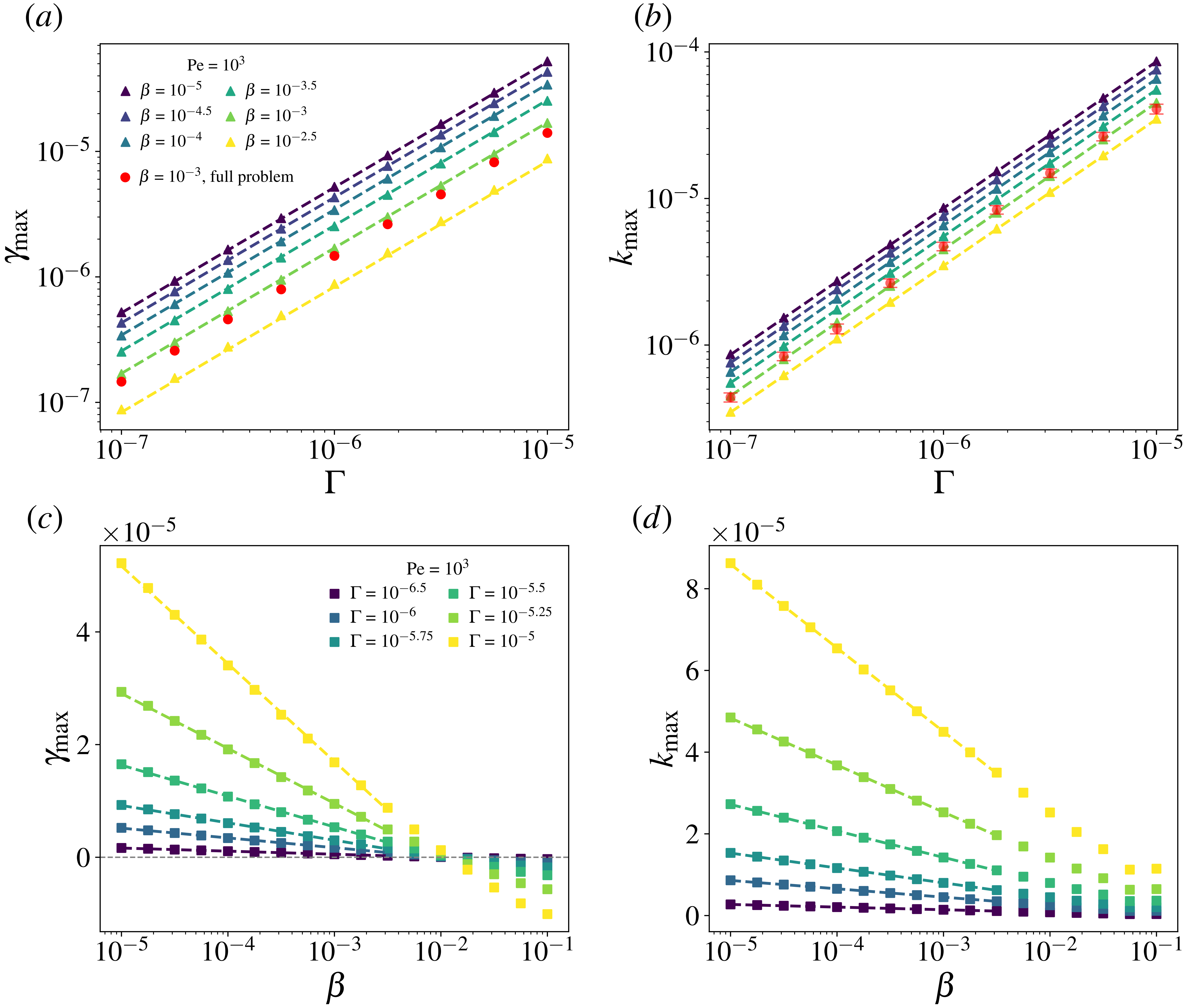
$$\gamma_{\max} = \gamma_{\max}(\beta, \Gamma, \mathrm{Pe})\ ?$$
$$k_{\max} = \gamma_{\max}(\beta, \Gamma, \mathrm{Pe})\ ?$$



Maximum growth rate and corresponding wavenumber

$\gamma_{\max}, k_{\max}$
vs Γ

$\gamma_{\max}, k_{\max}$
vs β



- In general:

$\Gamma \nearrow \implies \gamma_{\max} \nearrow, k_{\max} \nearrow$
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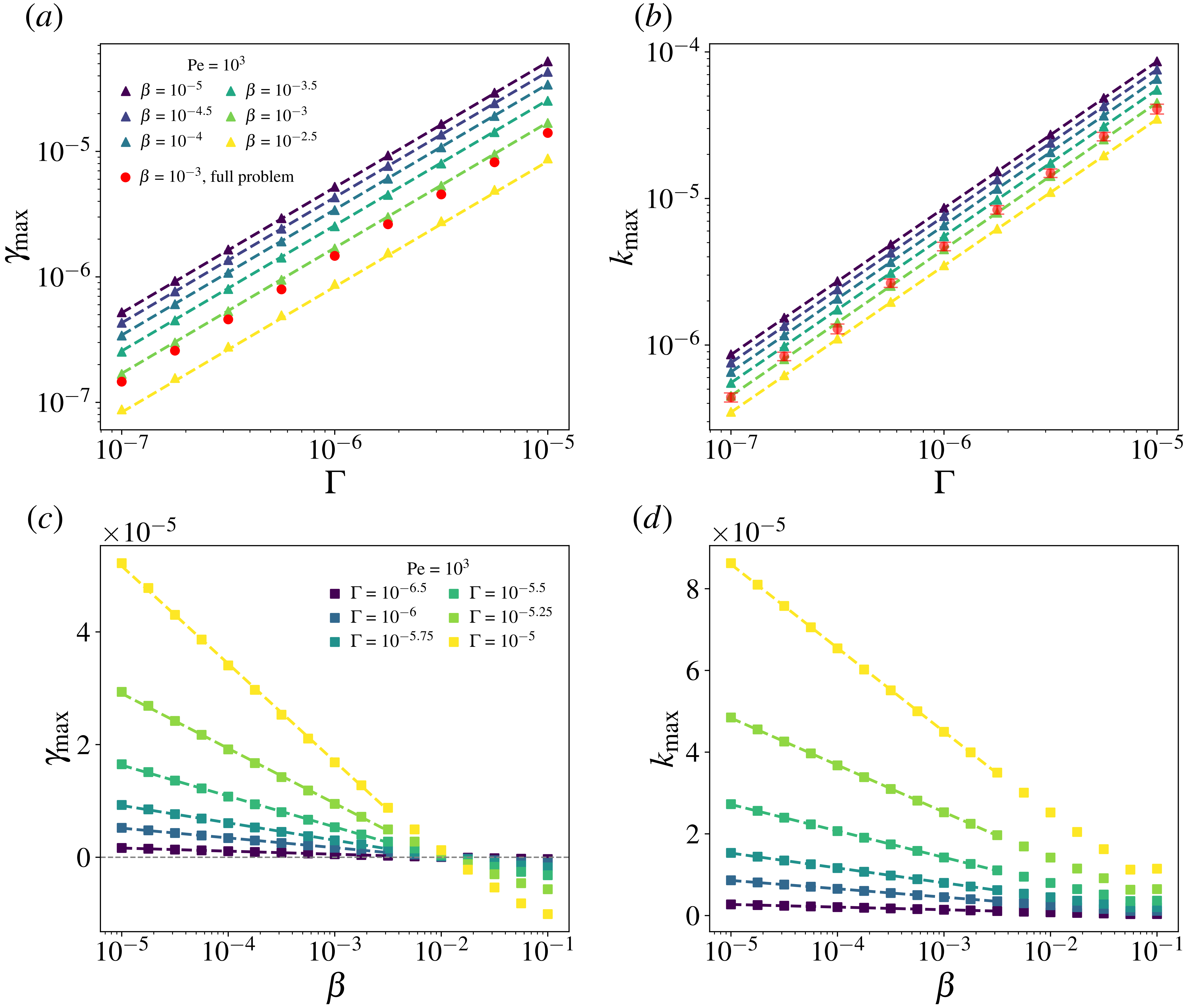

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- In the limit $\mathrm{Pe} \gg 1$ and $\beta \ll 1$:

$$\gamma_{\max} = \Gamma(a_\gamma \log \beta + b_\gamma)$$
$$k_{\max} = \Gamma(a_k \log \beta + b_k)$$

From fit:
$a_\gamma \approx -0.767 \pm 0.002,\ b_\gamma \approx -3.63 \pm 0.02$
$a_k \approx -3.63 \pm 0.02,\ b_k \approx -1.701 \pm 0.006$

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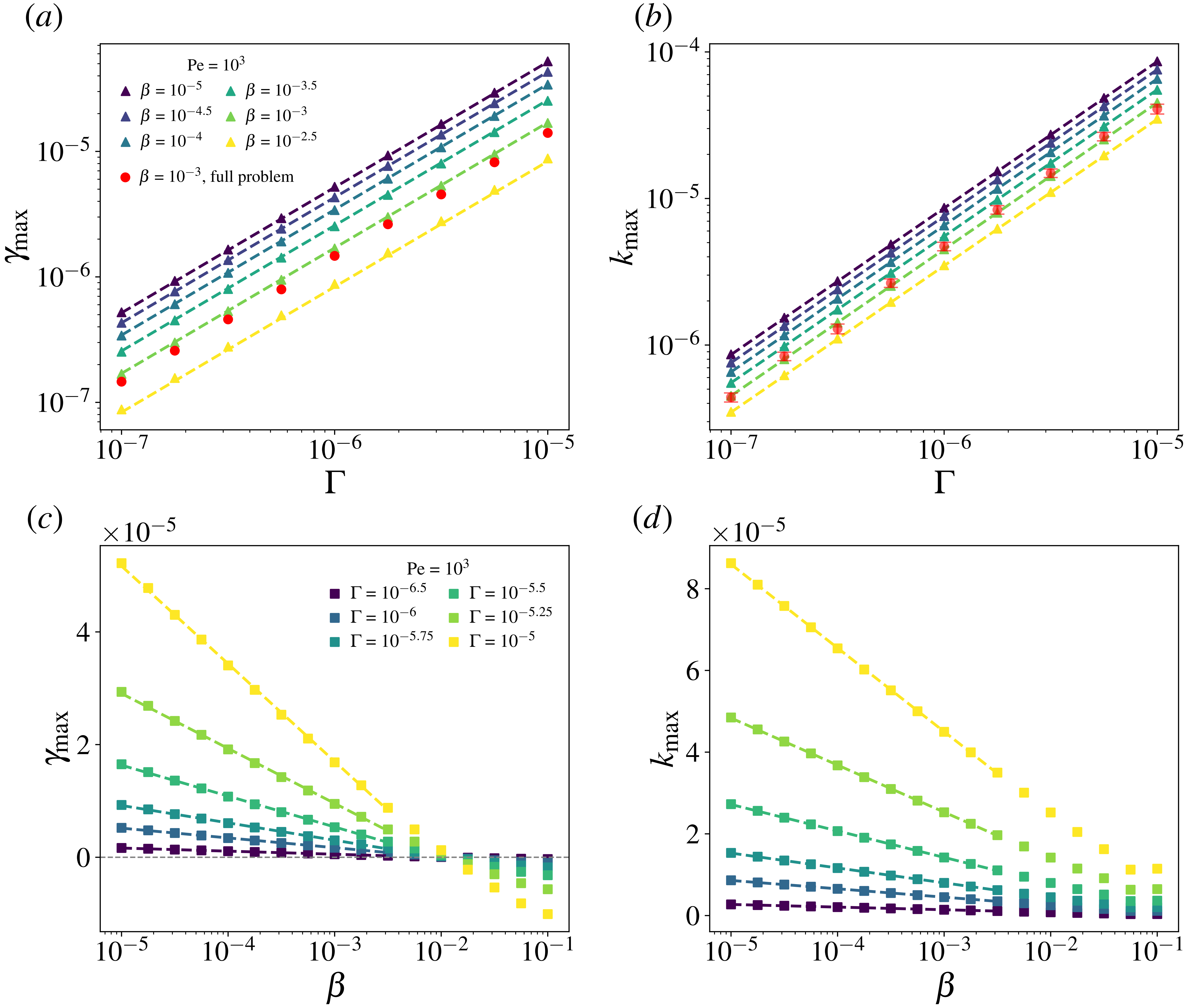

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- Critical viscosity ratio β_c:

$$\gamma_{\max} = 0 \iff \beta_c = e^{-b_\gamma/a_\gamma} \approx 0.0123$$

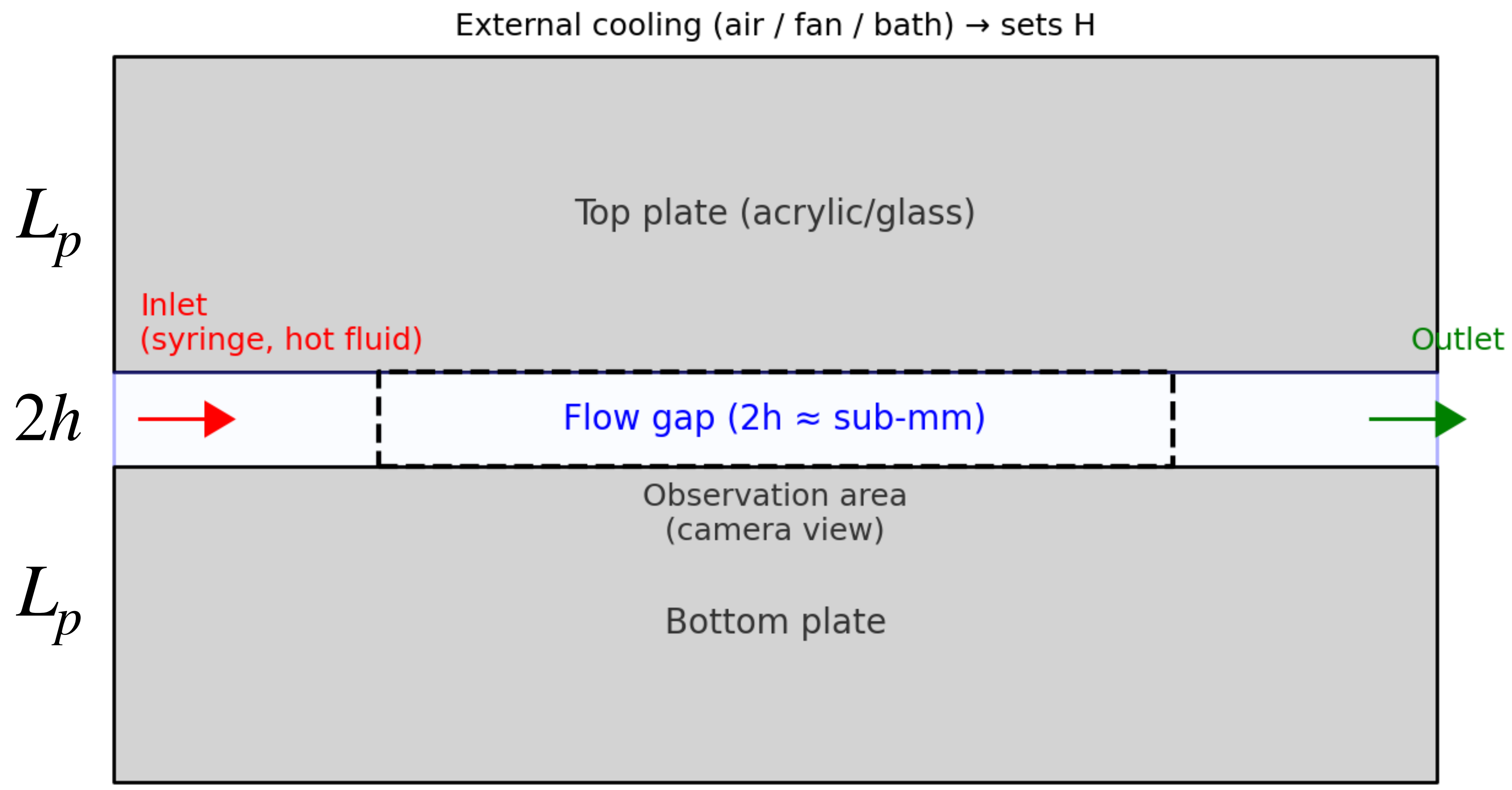
A possible lubrication experiment

Geometry: $h \sim 0.2 - 0.5\ \mathrm{mm}, L_p \sim 2 - 5\ \mathrm{cm}$

Glycerol: $\rho_f \approx 10^{-3}\,\mathrm{kg\,m^{-3}},\ c_f \approx 3 - 4 \times 10^3\ \mathrm{J\,kg^{-1}\,K^{-1}},\ \kappa_f \approx 10^{-6}\,\mathrm{m^2 s^{-1}}$

Inlet flow: $U \sim 10^{-4} - 10^{-3}\mathrm{m\,s^{-1}}$

Plate-flow heat transfer coefficient: $H_{\mathrm{ov}} \sim 20 - 200\,\mathrm{W\,m^{-2}\,K^{-1}}$



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Global parameters

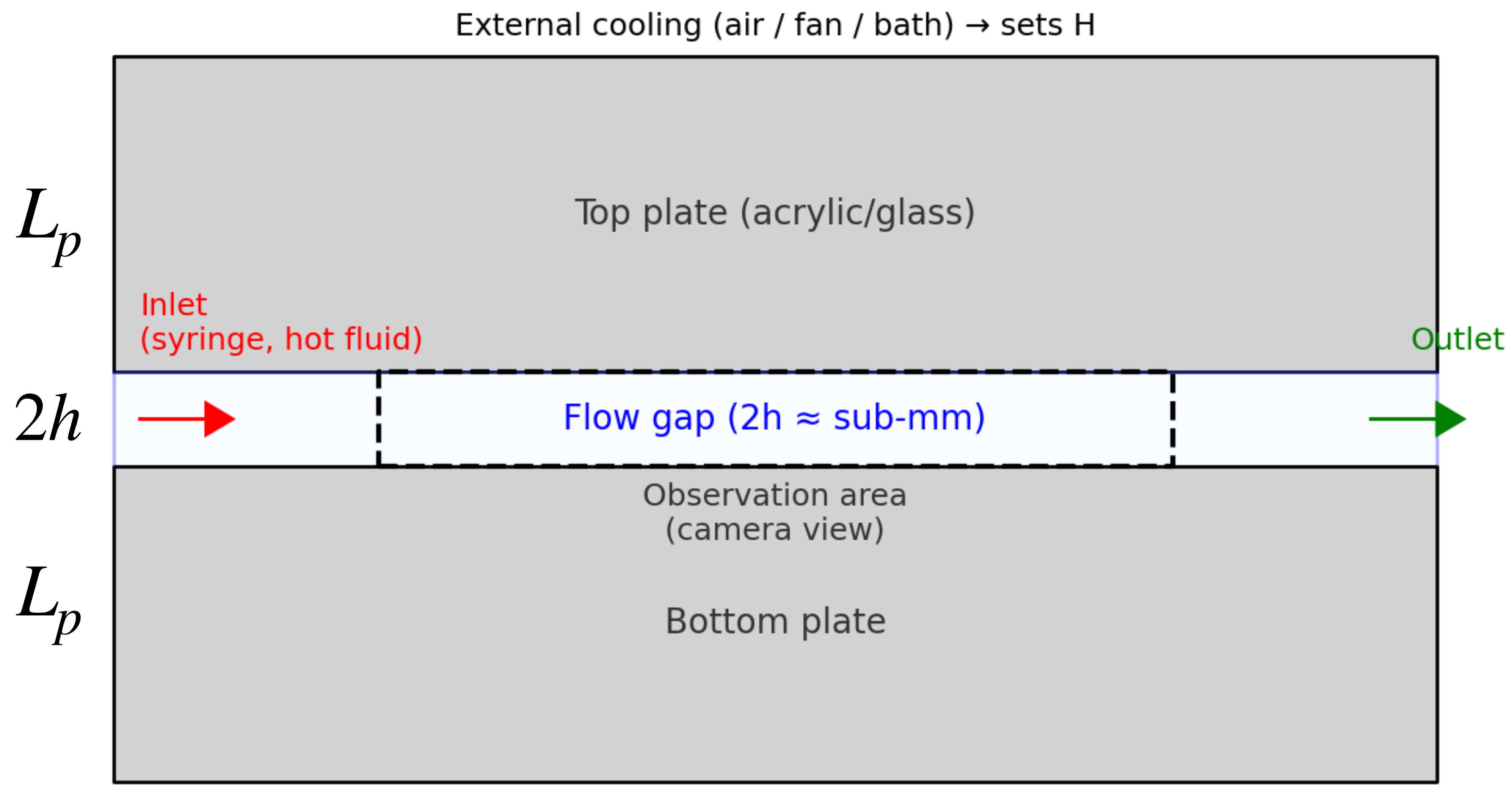
$$\beta = \frac{\mu(T_h)}{\mu(T_c)} \sim 10^{-3}\text{—}10^{-2}\,, \quad \Gamma = \frac{H_{\mathrm{ov}}}{\rho_f c_f U} \sim 10^{-2}\text{—}10^{-1}$$

$$\mathrm{Pe} = \frac{Uh}{\kappa_f} \sim 1\text{—}5 \quad (\mathrm{Bi} = \Gamma\,\mathrm{Pe} \sim 10^{-2} \ll 1)$$

Characteristic time and wavelength

$$t_{\mathrm{char}} = \frac{h}{U}\gamma_{\mathrm{max}}^{-1} \simeq \frac{\rho_f c_f h}{H_{\mathrm{ov}}} \frac{1}{a_\gamma \log\beta + b_\gamma} \simeq 10^2\text{—}10^3\,\mathrm{s}$$

$$\lambda_{\mathrm{char}} = 2\pi h\, k_{\mathrm{max}}^{-1} \simeq \frac{2\pi\rho_f c_f h U}{H} \frac{1}{a_k \log\beta + b_k} \sim 5\text{—}20\,\mathrm{mm}$$

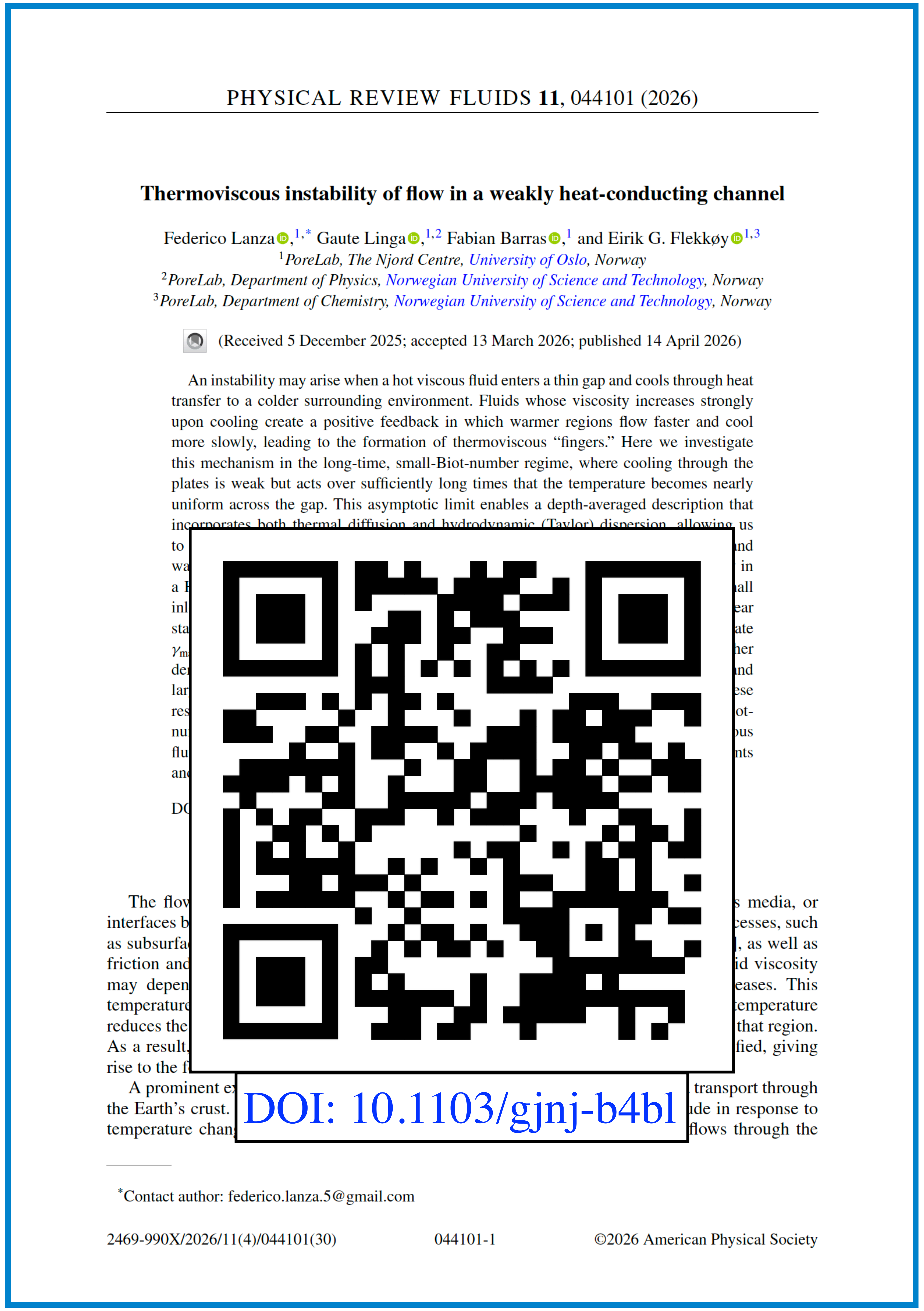


Conclusions

- Numerical simulation of invasion of **temperature-dependent viscous fluid** in a Hele-Shaw cell geometry
- **Dynamic instability** reproduced in a region of the $(\beta, \Gamma, \mathrm{Pe})$ phase space
- Linear Stability Analysis: dispersion relationship and behavior of the **maximum growth rate** $\gamma_{\max}$ **and wavelength** $k_{\max}$

➡ Comparison with experiments? (thermal viscous fluids in small gaps)

Thank you for your attention!



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Thermoviscous instability of flow in a weakly heat-conducting channel

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An instability may arise when a hot viscous fluid enters a thin gap and cools through heat transfer to a colder surrounding environment. Fluids whose viscosity increases strongly upon cooling create a positive feedback in which warmer regions flow faster and cool more slowly, leading to the formation of thermoviscous "fingers." Here we investigate this mechanism in the long-time, small-Biot-number regime, where cooling through the plates is weak but acts over sufficiently long times that the temperature becomes nearly uniform across the gap. This asymptotic limit enables a depth-averaged description that incorporates both thermal diffusion and hydrodynamic (Taylor) dispersion, allowing us

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