Shearlet, a Novel Operator Learning Model

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Motivation: learning solution operators of parametric PDEs

Problem statement

Given a parametric PDE

$$\mathcal{N}(u; a) = 0, \qquad a \in \mathcal{A},\ u \in \mathcal{U},$$

learn the solution operator $\mathcal{G}^\dagger : a \mapsto u$ from data, so that new instances of a can be mapped to u without re-running a classical solver.

Why does this matter?

- Many-query problems: uncertainty quantification, design, inverse problems, control.
- Classical solvers are accurate but expensive when repeatedly evaluated.
- Neural operators trade offline training cost for fast online inference.

The catch. Real PDE solutions often exhibit:

- Anisotropic structures aligned with characteristics or material directions.
- Sharp gradients, fronts, and shocks.
- Multi-scale interactions.

Neural operators in a nutshell

Goal. Approximate $\mathcal{G}^\dagger$ between function spaces, in a discretization-invariant way.

A typical neural-operator layer acts on a hidden function $v_t(x)$ as

$$v_{t+1}(x) = \sigma\Big(W\, v_t(x) \;+\; \big[\mathcal{K}(a)\, v_t\big](x)\Big),$$

combining a local linear map W with a non-local integral operator

$$\big[\mathcal{K}(a)\, v\big](x) \;=\; \int_D \kappa(x, y; a)\, v(y)\, dy.$$

Design question: how do we parameterise / compute $\mathcal{K}$ efficiently?

- Graph kernels (GNO): flexible but costly.
- Low-rank kernels (LNO): efficient but global.
- Spectral parameterisation $\Rightarrow$ FNO and its descendants.

Fourier Neural Operator (FNO)

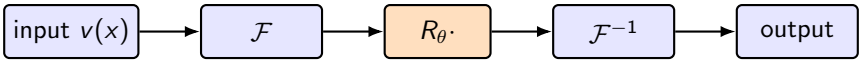
Key idea. Parameterise the integral kernel *in the spectral domain*: convolution becomes pointwise multiplication.

$$(\mathcal{K}\, v)(x) \;=\; \mathcal{F}^{-1}\Big(R_\theta \cdot \mathcal{F}(v)\Big)(x),$$

where $\mathcal{F}$ is the Fourier transform and R_θ is a learnable, frequency-truncated multiplier.

Strengths

Fast (via FFT), mesh-agnostic, strong on smooth and periodic problems.



Where FNO struggles

The Fourier basis $\{e^{i\langle k,x\rangle}\}$ is:

- Global: every basis function is supported everywhere.
- Isotropic: no preferred direction.
- Smooth: poor at representing discontinuities (Gibbs phenomenon).

Consequence. For PDEs with

- anisotropic structure (e.g. strongly advective transport),
- shocks and curved fronts (conservation laws),
- sharp interfaces (multi-phase flows, multi-scale media),

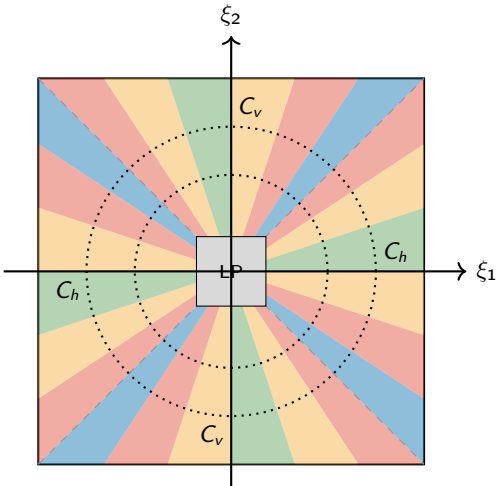
FNO requires many modes and still smears out fine localized features.

What we want

A spectral representation that is directional, multiscale, and spatially localized.
$\Longrightarrow$ **Shearlets**.

Background — Shearlets: the intuition

Cartoon-like signals (images, PDE solutions): piecewise smooth with discontinuities along curves. We want a representation that is multiscale, directional, and spatially localised.



Frequency-plane tiling.

- Two cones — C_h (horizontal) and C_v (vertical) — plus a central low-pass square.
- Each cone is split into dyadic radial shells indexed by scale j.
- Each shell is partitioned into angular wedges indexed by shear k.
- Finer scales $\Rightarrow$ more wedges $\Rightarrow$ better directional resolution.

Three group operations:

- A_a parabolic scaling
- S_s shearing (orientation)
- T_t translation (locality)

Background — Why shearlets are powerful

Optimal sparse approximation (Guo & Labate)

For the class $\mathcal{E}^2(\mathbb{R}^2)$ of *cartoon-like* functions (piecewise C^2 with C^2 edges), the best N-term shearlet approximation f_N satisfies

$$\|f - f_N\|_2^2 \lesssim N^{-2} (\log N)^3,$$

which is optimal up to a log factor, better than Fourier or wavelets.

Three crucial properties:

- Directional selectivity: shear parameter resolves orientation, ideal for fronts and anisotropy.
- Parabolic scaling $a \times \sqrt{a}$: matches the geometry of curvilinear singularities.
- Spatial localisation: unlike Fourier atoms, each shearlet is concentrated around t.

These are exactly the features needed for shock-dominated PDE solutions.

Background — Shearlets vs. Fourier vs. Wavelets

Property	**Fourier**	**Wavelets**	**Shearlets**
Spatial localisation	No	Yes	Yes
Multiscale	No	Yes	Yes
Directional selectivity	No	Limited	Yes
Anisotropic atoms	No	No	Yes
Sparse approx. of curved edges	$N^{-1/2}$	N^{-1}	$N^{-2}(\log N)^3$
Fast transform available	FFT	FWT	Fast Shearlet Transform

Take-away

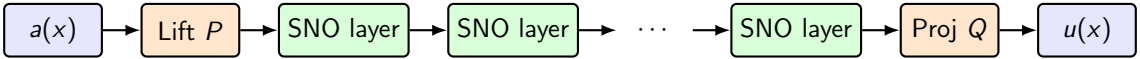
Shearlets keep the spectral-style efficiency that makes FNO attractive, while supplying the geometric inductive bias that Fourier lacks.

Shearlet Neural Operator (SNO) — architecture

Idea. Replace the Fourier transform in the spectral layer of FNO with the discrete shearlet transform SH.

A single SNO layer applied to a hidden representation v_t:

$$v_{t+1}(x) \;=\; \sigma\Big(W\, v_t(x) \;+\; \mathrm{SH}^{-1}\big(R_\theta \odot \mathrm{SH}(v_t)\big)(x)\Big).$$



Pipeline. Lifting $P \to$ stacked SNO layers $\to$ projection Q to physical output.

Benchmark suite — seven PDE families

We compare SNO against FNO on seven 2D parametric problems grouped by regime:

Diffusion-dominated

- **Multi-orientation texture** — anisotropic diffusion of superposed oriented waves.

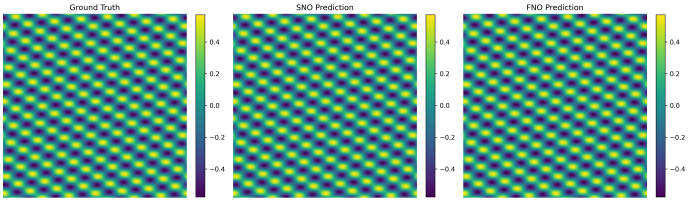
Convection-dominated

- **Bent ridge advection** — parabolically curved front.
- **Anisotropic ridge advection** — extreme aspect-ratio jet ($c_y/c_x = 120$).
- **Sheared Kelvin–Helmholtz stripes** — noisy oriented stripes.
- **Polygonal shock pattern** — 6-fold angular symmetry.

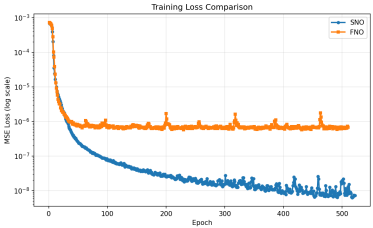
Combined-effect (Burgers, low viscosity)

- **Multi-angle shocks** — two interacting shock fronts.
- **Spiral shock** — Archimedean spiral with continuously varying orientation.

Results — Multi-orientation texture (diffusion)



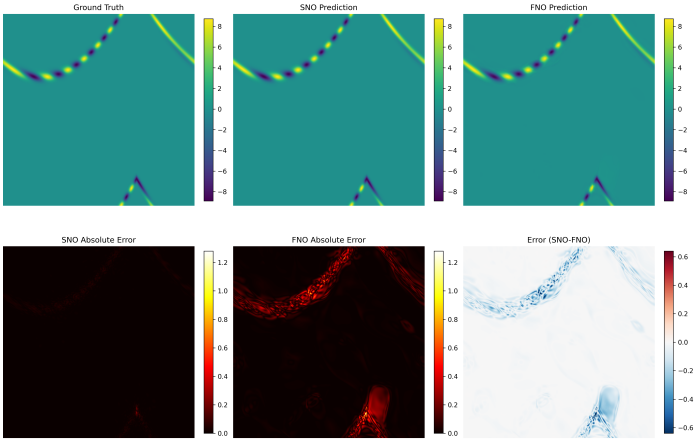
Anisotropic diffusion of superposed oriented waves. (N=512, T=150)



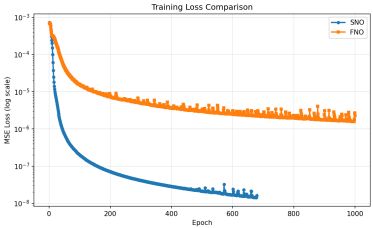
Training loss (MSE, log).

- FNO error visibly larger inside the oriented bands; SNO recovers them sharper.
- Both operators capture the multi-scale texture; L^2 ratio ≈ 0.97 — near-tie at this resolution.
- FNO loss plateaus earlier; SNO converges faster in the early epochs.

Results — Bent ridge advection



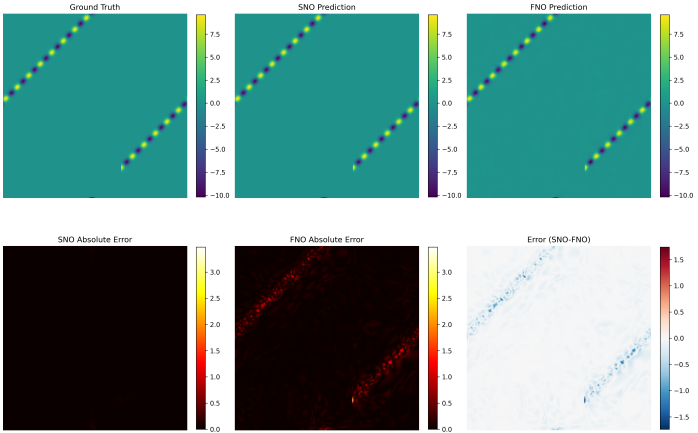
Curved front advected with $(c_x, c_y) = (0.5, 5)$. (N=512, T=150)



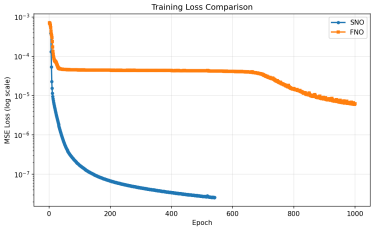
Training loss (MSE, log).

- SNO error near zero along the curved trajectory; FNO smears it over a wide tube.
- L^2 ratio (SNO/FNO) ≈ 0.10 — an order of magnitude better.
- Loss gap is roughly two orders of magnitude throughout training.

Results — Anisotropic ridge advection (jet stream)



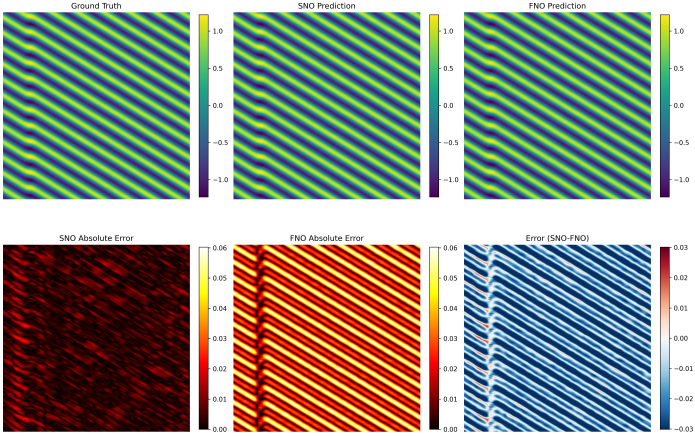
Aspect-ratio 120 transport, $(c_x, c_y) = (0.05, 6.0)$. (N=512, T=150)



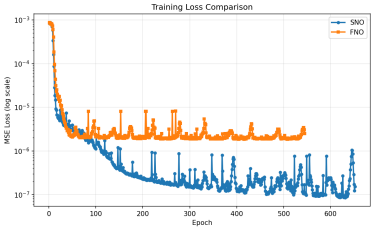
Training loss (MSE, log).

- Extreme directional bias: FNO's isotropic modes cannot align with the jet.
- SNO captures the oriented ridge essentially exactly.
- Largest gap of the suite: L^2 ratio ≈ 0.06 ($\sim$16$\times$); SSIM 0.9999 vs. 0.958.

Results — Sheared Kelvin–Helmholtz stripes



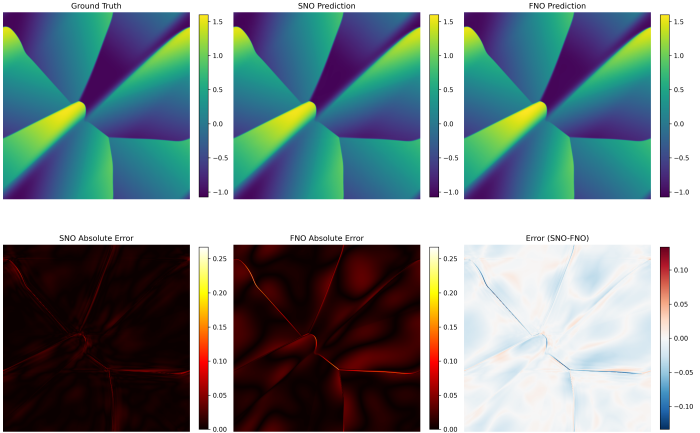
Sheared stripes with additive Gaussian noise. (N=512, T=100)



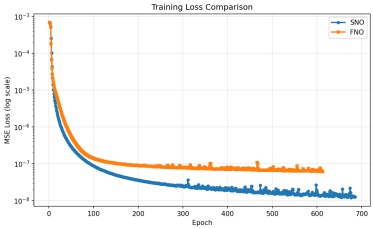
Training loss (MSE, log).

- Noise stresses both operators, but SNO retains stripe orientation and contrast.
- FNO loses stripe sharpness; SSIM drops to 0.968 vs. 0.999 for SNO.
- L^2 ratio ≈ 0.18 — consistent 5–6$\times$ improvement.

Results — Polygonal shock pattern



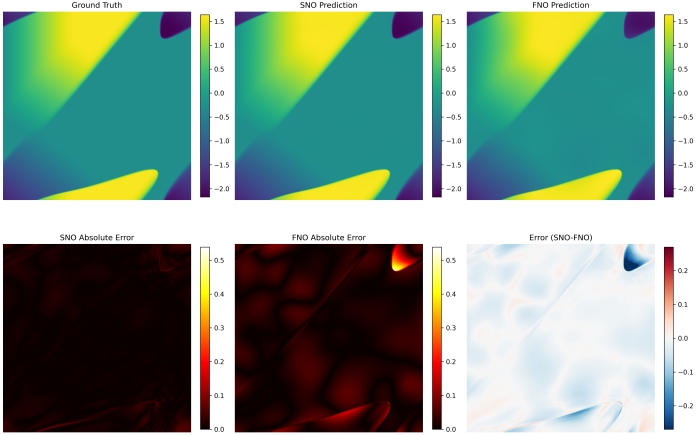
Hexagonal shock with 6-fold symmetry. (N=512, T=220)



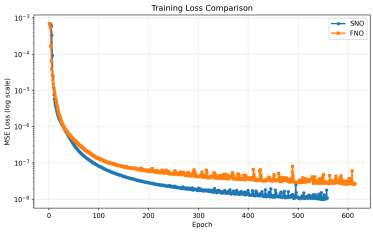
Training loss (MSE, log).

- FNO rounds the polygon's corners; SNO preserves the sharp vertices.
- Error concentrates on the discrete symmetry axes.
- L^2 ratio ≈ 0.43 — gap narrows because structure aligns partly with grid.

Results — Multi-angle shocks (Burgers)



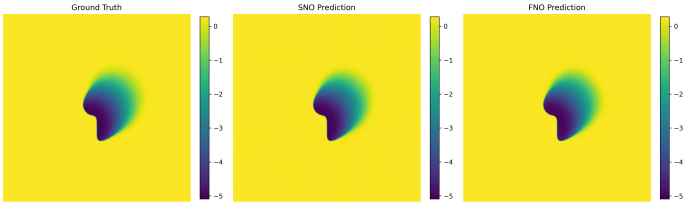
Two interacting shock fronts at $\arctan(0.8)$ and $\arctan(-1.2)$. (N=512, T=220)



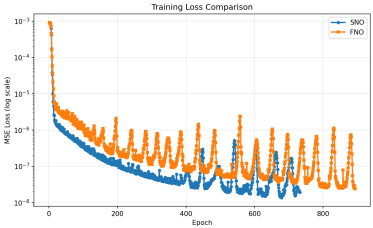
Training loss (MSE, log).

- SNO is visually indistinguishable from ground truth across the domain.
- FNO blurs both fronts and introduces oscillations near intersections.

Results — Spiral shock pattern (Burgers)



Archimedean spiral with continuously varying orientation. (N=512, T=100)



Training loss (MSE, log).

- Continuously rotating front: the hardest case for a finite shear discretisation.
- SNO still wins overall; FNO slightly better in smooth interior regions (red/blue mix).
- L^2 ratio ≈ 0.99 — essentially tied; limiting case for SNO's bias.

Quantitative comparison — all benchmarks (N=512)

Table: Performance Comparison between SNO and FNO Models for N = 512

Dataset	SNO L2	FNO L2	Ratio	SNO MSE	FNO MSE	SNO MAE	FNO MAE	SNO SSIM	FNO SSIM
anisotropic ridge advect	2.41×10^{-6}	3.97×10^{-5}	0.0608	3.36×10^{-5}	9.08×10^{-3}	2.11×10^{-3}	4.60×10^{-2}	0.9999	0.9583
multi angle shocks	3.93×10^{-6}	1.48×10^{-5}	0.2650	9.08×10^{-5}	1.29×10^{-3}	6.25×10^{-3}	2.17×10^{-2}	0.9990	0.9942
bent ridge advect	2.53×10^{-6}	2.52×10^{-5}	0.1004	3.69×10^{-5}	3.67×10^{-3}	2.43×10^{-3}	2.24×10^{-2}	0.9999	0.9780
multi orientation texture	4.09×10^{-6}	4.21×10^{-6}	0.9699	8.38×10^{-6}	8.91×10^{-6}	2.25×10^{-3}	2.46×10^{-3}	0.9986	0.9988
spiral shock	8.36×10^{-6}	8.49×10^{-6}	0.9852	2.16×10^{-4}	2.22×10^{-4}	5.84×10^{-3}	4.42×10^{-3}	0.9939	0.9942
polygonal shock	3.52×10^{-6}	8.18×10^{-6}	0.4308	4.10×10^{-5}	2.21×10^{-4}	4.19×10^{-3}	1.04×10^{-2}	0.9987	0.9934
sheared kelvin helmotz	4.18×10^{-6}	2.31×10^{-5}	0.1812	3.21×10^{-5}	9.77×10^{-4}	4.44×10^{-3}	2.65×10^{-2}	0.9989	0.9677

- Gap grows with anisotropy and shock complexity; spiral shock and multi-orientation texture are the limiting near-tie cases.

Discussion

What changed? Only the spectral basis — from Fourier exponentials to shearlet atoms. Architecture, training, data, parameter budget were held fixed (SNO actually uses *fewer* parameters: $\sim$12k vs. $\sim$17k for FNO).

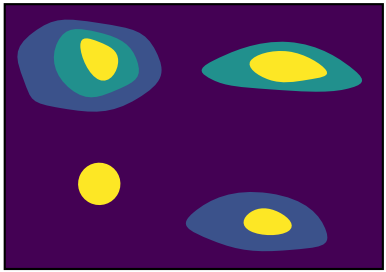
Why such a large effect on hard regimes?

- Fourier needs many modes to approximate localised, anisotropic features.
- Shearlets are near-optimal for cartoon-like functions $\Rightarrow$ a small number of coefficients carries most of the information.
- The neural network only has to learn the residual structure once the basis already captures the geometry.

Limitations.

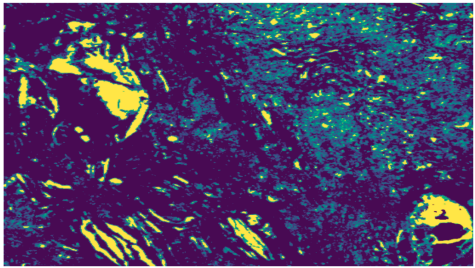
- Shearlet transforms have a higher constant cost than FFT.
- Cases with continuously varying orientation (spiral shock) erode the inductive-bias advantage.
- 3D extensions exist but are heavier to implement and train.

Bridge — carbonate porosity slices *exhibit* cartoon-like structure



Synthetic cartoon $f \in \mathcal{E}^2(\mathbb{R}^2)$:

piecewise smooth on C^2 regions; sharp jumps on a C^2 curve.



Real porosity field (μCT):

Near-constant interiors, curved pore–solid interfaces.

Why this matters

Similar structure: roughly constant interiors with sharp jumps along curved boundaries. The Guo–Labate $N^{-2}(\log N)^3$ rate (slide 9) is optimal *exactly* for this signal class — a stronger fit than "anisotropic" alone.

Application — Porosity prediction from dry rock samples

Setting. 3D porosity mapping of carbonate plug samples via micro-CT (COPPE/UFRJ × Petrobras).

Classical pipeline (paired dry/wet μCT)

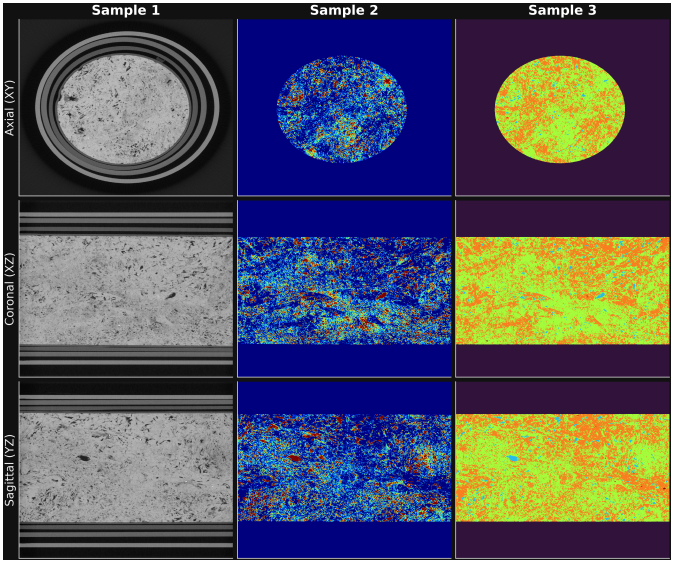
- Two scans of the *same* plug: dry and saturated with 10% NaI at 2,500 psi.
- Calibrated intensity difference gives voxel-wise porosity.
- Accurate but expensive: two scans, chemicals, high-pressure cel.

Goal. Skip the wet scan — predict the porosity map $\phi(x) \in [0,1]$ directly from the dry image alone, at *voxel scale*.

Why is SNO a natural fit here?

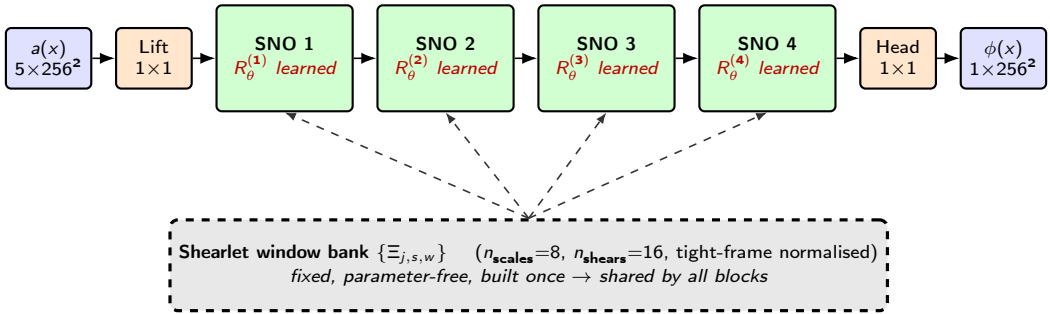
- Carbonate pore networks are strongly anisotropic (vugs, channels, oolitic shells, laminations).
- Pore-solid interfaces are near-cartoon: piecewise-smooth fields with curved C^2 jump sets — the regime where shearlets give near-optimal sparse approximation.
- Dataset: 11 plugs.

Application — Porosity prediction from dry rock samples



Spectral core — shared bank, per-block learned weights

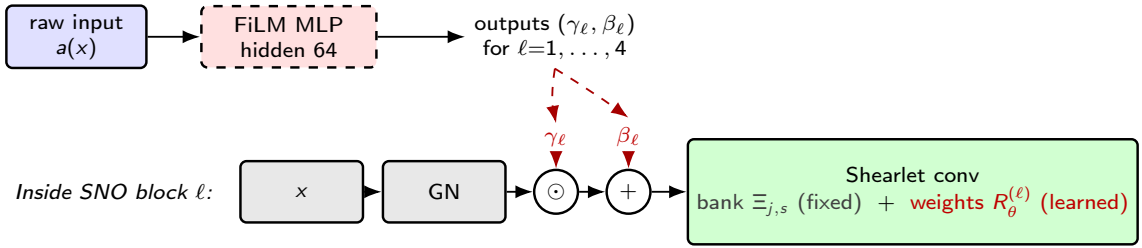
Task. Input: $5\times256\times256$ (5 neighbouring dry slices). Output: $1\times256\times256$ (central porosity slice). $L=4$ spectral layers between a lift and a head.



Inside each SNO block, *all of these are learned, separately per block*: spectral mixing weights $R_\theta^{(\ell)}$ acting on shearlet coefficients, sigmoid gates that combine the three mixing branches (spectral + depthwise 3×3 + pointwise 1×1), the depthwise/pointwise conv weights themselves, SE channel attention ($r=4$), and the transformer-style FFN ($4\times$ expansion). Channel width $W=16$ throughout. Per-sample GroupNorm ($G=1$); long-range skip from lifted input to head.

FiLM conditioning — how the *fixed* bank adapts per sample

Feature-wise Linear Modulation. A small MLP reads the raw input a and emits sample-specific scale/shift pairs $\mathrm{FiLM}(x;a) = \gamma(a) \odot x + \beta(a)$, with $(\gamma_\ell, \beta_\ell) \in \mathbb{R}^W \times \mathbb{R}^W$ for each of the L=4 blocks (W=16 values each).



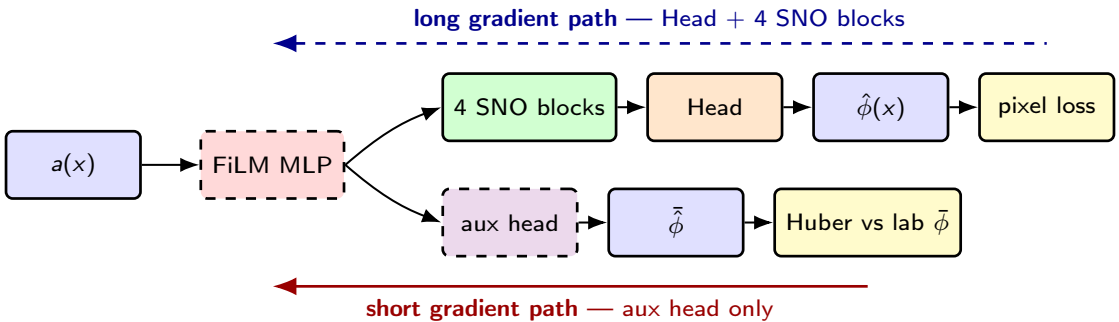
Shared across samples (set once during training):

- the shearlet bank $\Xi_{j,s,w}$ (fixed mathematics, parameter-free);
- every learned weight: $R_\theta^{(\ell)}$, FiLM-MLP, sigmoid gates, depthwise/pointwise convs, SE, FFN.

Sample-specific (depend on *this* $a(x)$):

- only the values $(\gamma_\ell, \beta_\ell)$;
- γ_ℓ rescales each channel before the shearlet conv $\Rightarrow$ the fixed bank Ξ acts as an effectively different filter on each input, without any weight changing.
- Zero-init (γ=1, β=0 at step 0): training starts as if FiLM is absent.

Auxiliary bulk-porosity head — a shortcut for global calibration



What the aux head does. A small MLP off the FiLM encoder predicts *one number* per sample — the bulk porosity:

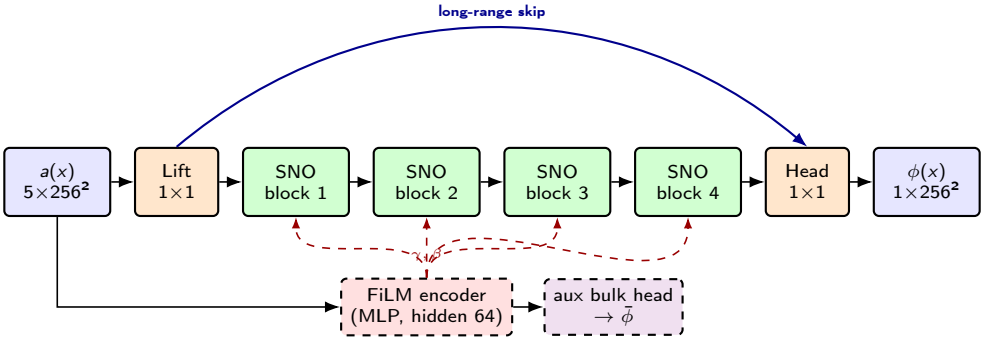
$$\bar{\hat{\phi}}(a) \;=\; \mathrm{AuxHead}\big(\text{FiLM-MLP}(a)\big),$$

trained against the lab value with a Huber loss (weight 0.5).

Why this matters.

- Bulk porosity is a sample-level property; via pixel loss alone, calibration must back-propagate through 4 spectral blocks + head before reaching FiLM.
- Aux loss at the FiLM encoder gives a one-step gradient path $\Rightarrow$ FiLM learns bulk *early*, so the spectral core can specialise on *spatial structure*.

SNO for porosity — architecture diagram



Detail of one SNO block (↓):

Training protocol — optimisation & LOOCV setup

Optimisation

- AdamW, $\beta = (0.9, 0.95)$
- $\eta = 10^{-3}$; weight decay 10^{-4}
- Batch size 32; gradient clip 2.0
- Up to 1000 epochs; early-stop patience 50 on `val_loss`
- GroupNorm-based per-sample input normalisation; no batch statistics

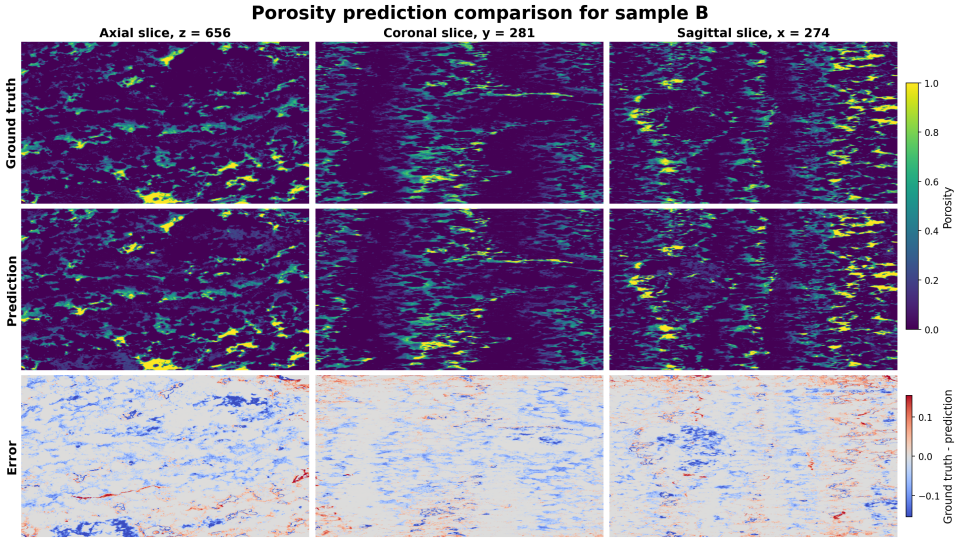
LOOCV setup

- 11 sample codes, one held out per fold ($\times$11 retrains)
- 90 / 10 train / val split within each fold
- Sqrt-weighted sample balancing (per-sample voxel count)
- DDP, multi-GPU; resumes from `last.ckpt`

Reporting

Test metrics are aggregated across the 11 held-out folds. Per-fold bulk porosities are then compared against laboratory measurements.

Application results — visual prediction quality



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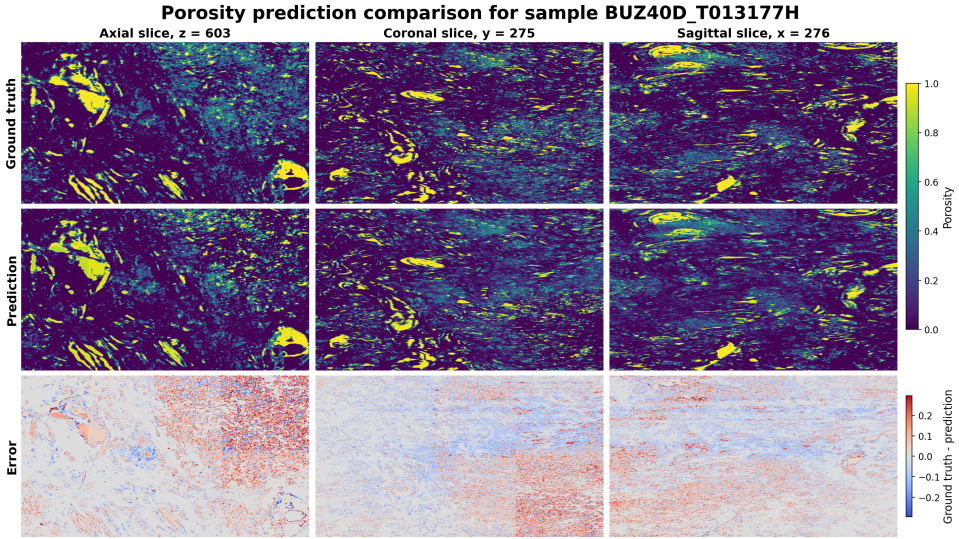
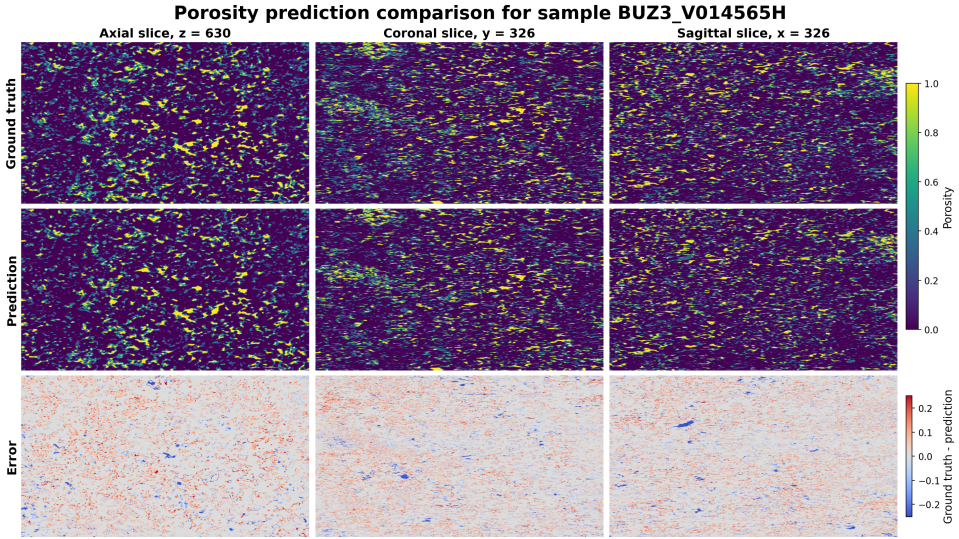
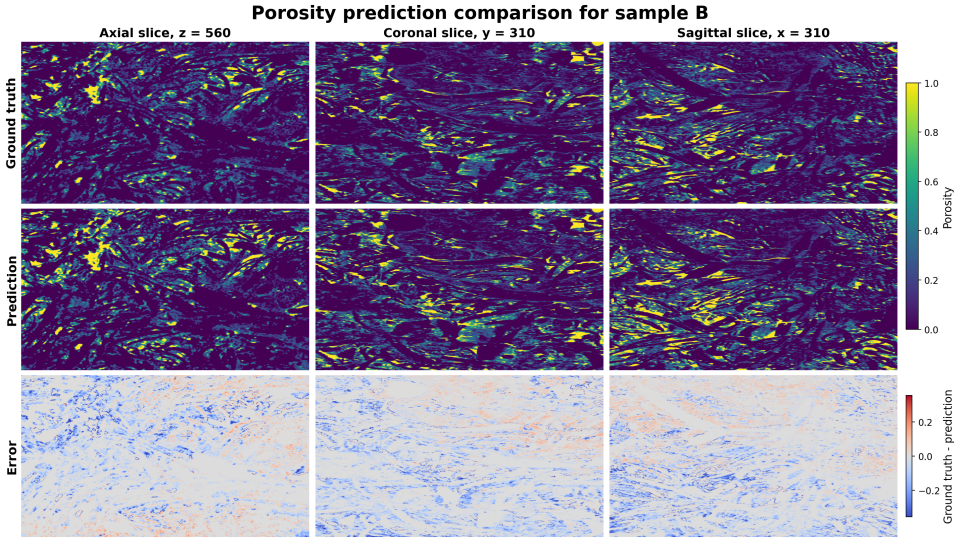


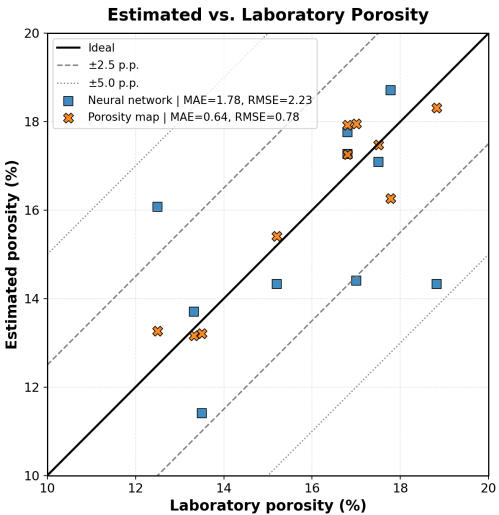
Application results — visual prediction quality



Application results — visual prediction quality



Application results — bulk porosity vs. classical CDF



Predicted vs. lab bulk porosity, 11 plug crops, LOOCV.

Metric (p.p. unless noted)	Porosity map	SNO
MAE	0.64	1.78
RMSE	0.78	2.23
Bias	+0.18	−0.63
Median AE	0.51	0.96
Max AE	1.52	4.49
MAPE (%)	3.80	10.95
R^2	0.91	0.28
Within ±1 p.p. (%)	72.73	54.55
Within ±2.5 p.p. (%)	100.00	63.64
Within ±5 p.p. (%)	100.00	100.00

- SNO recovers spatial structure of pores per-slice (see prev. slide).
- Bulk calibration trails the porosity map method, which uses *both* dry+wet scans by construction.
- All SNO predictions remain within ±5 p.p. of lab porosity — with the wet scan dropped.

Conclusions, outlook & references

Contributions

- **SNO:** a neural operator built on the discrete shearlet transform — directional, multiscale, spatially-localised spectral layer.
- Consistent improvements over FNO on anisotropic and shock-dominated parametric PDEs across seven benchmark families, with *fewer* parameters.
- Real-world case study: 3D porosity mapping of carbonate rocks from dry-state micro-CT. SNO recovers pore geometry across 11 LOOCV folds; all bulk-porosity predictions within ± 5 p.p., *without* the saturated scan.

Outlook

- 3D shearlets for compressible flow, seismic imaging, and full-volume porosity prediction.
- Hybrid Fourier / shearlet layers; physics-informed losses; adaptive band truncation.
- Tighten bulk-porosity calibration on the rock dataset (mass-conservation loss; larger / more diverse training cores).

Preprint

F. P. dos Santos, J. de C. V. Fernandes, A. M. A. Côrtes. *Shearlet Neural Operators for Anisotropic-Shock-Dominated and Multi-scale Parametric PDEs.* arXiv:2604.25181 — arxiv.org/abs/2604.25181

Thank you — questions?