Energy-Preserving TPFA Scheme for Compressible Gas Flow in Deformable Porous Media

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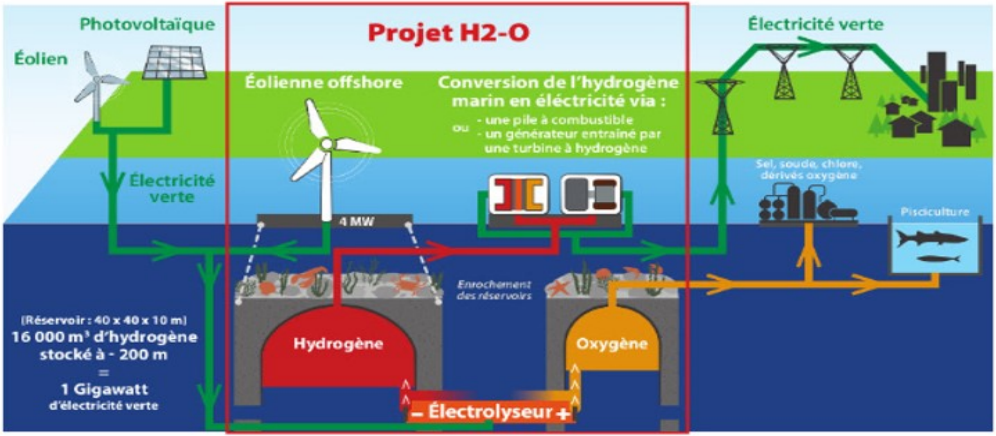
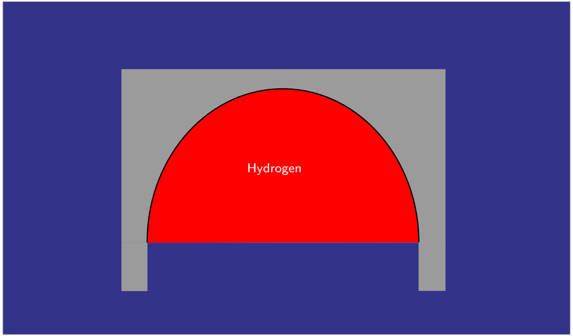
Introduction

Motivation



- ➔ As **offshore wind farms** expand, so do concerns about managing excess **electricity** during peak production.
- ➔ A promising solution is to convert surplus **energy** into **hydrogen** via **water electrolysis**, then store it in underwater **cementitious cavities**.

- Chemical degradation.
- Structural damage.
- Loss of mechanical strength.
- Increased leak risk.



Underwater cementious cavity

Objective: Use mathematical models and numerical simulations to predict and prevent these issues, ensuring safe, efficient, and long-term hydrogen storage.

Mathematical model

Thermo-Hydro-Mechanical model

We consider a **Thermo-Hydro-Mechanical** (THM) model[1,2] describing a single phase **compressible** flow in a porous medium.

[1] O. Coussy. *Poromechanics*. Wiley, 2004.

[2] Jérôme Droniou, Mohamed Laaziri, and Roland Masson. "Thermodynamically Consistent discretisation of a Thermo-Hydro-Mechanical model". In: *Springer Proceedings in Mathematics and Statistics* (2023).

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Mass conservation equation:

$$\partial_t(\rho(p)\phi(\boldsymbol{u}, p, T)) + \mathrm{div}(\rho(p)\boldsymbol{V}(p)) = h_m.$$

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Entropy conservation equation:

$$\partial_t(S_s(\boldsymbol{u},p,T) + \rho(p)\phi(\boldsymbol{u},p,T)s) + \mathrm{div}\left(\rho(p)s\boldsymbol{V}(p) + \frac{1}{T_{\mathrm{ref}}}\boldsymbol{q}(T)\right) = \frac{h_e}{T}.$$

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Momentum balance equation:

$$m_0\partial_{tt}^2\boldsymbol{u} - \mathrm{div}(\boldsymbol{\sigma}(\boldsymbol{u},p,T)) = \boldsymbol{h}.$$

The main unknowns are the fluid pressure (p), the fluid temperature (T) and the displacement of the skeleton ($\boldsymbol{u}$).

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Darcy law:

$$\boldsymbol{V}(p) := -\frac{\mathbb{K}}{\mu}\left(\boldsymbol{\nabla} p - \rho(p)\mathrm{g}\boldsymbol{\nabla}\boldsymbol{z}\right).$$

Fourier law:

$$\boldsymbol{q}(T) := -\lambda\boldsymbol{\nabla} T.$$

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Linear isotropic thermo-poro-elastic constitutive laws:

$$\boldsymbol{\sigma}(\boldsymbol{u}, p, T) := \boldsymbol{\sigma}^e(\boldsymbol{u}) - bp\mathbb{I}_d - 3\alpha_s K_s (T - T_{\text{ref}})\mathbb{I}_d,$$

$$\boldsymbol{\sigma}^e(\boldsymbol{u}) := \frac{\mathcal{E}}{1+\nu}\left(\boldsymbol{\epsilon}(\boldsymbol{u}) + \frac{\nu}{1-2\nu}\mathrm{div}\boldsymbol{u}\mathbb{I}_d\right),$$

$$\partial_t \phi(\boldsymbol{u}, p, T) = b\partial_t(\mathrm{div}\boldsymbol{u}) - 3\alpha_\phi \partial_t T + \frac{1}{N}\partial_t p,$$

$$\partial_t S_s(\boldsymbol{u}, p, T) = 3\alpha_s K_s \partial_t(\mathrm{div}\boldsymbol{u}) - 3\alpha_\phi \partial_t p + \frac{C_s}{T_{\text{ref}}}\partial_t T.$$

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\partial_t S_s(\boldsymbol{u}, p, T) &= 3\alpha_s K_s \partial_t(\mathrm{div}\boldsymbol{u}) - 3\alpha_\phi \partial_t p + \frac{C_s}{T_{\mathrm{ref}}}\partial_t T.
\end{aligned}$$

Boundary conditions:

$$\begin{aligned}
\boldsymbol{\nabla} p \cdot \boldsymbol{n} = 0 &\quad \text{in} \quad \partial\Omega \times (0, t_F), \\
\boldsymbol{\nabla} T \cdot \boldsymbol{n} = 0 &\quad \text{in} \quad \partial\Omega \times (0, t_F), \\
\boldsymbol{u} = 0 &\quad \text{in} \quad \partial\Omega \times (0, t_F).
\end{aligned}$$

Mathematical Resolution

- **Energy estimates for the continuous model:**

$$\|p\|_{L^2(0,t_F;H^1(\Omega))} + \|T\|_{L^2(0,t_F;H^1(\Omega))} + \|\boldsymbol{u}\|_{L^2(0,t_F;H_0^1(\Omega))^d} \leq C.$$

[3] Robert Eymard, Thierry Gallouët, and Raphaèle Herbin. "Finite volume methods". In: *Handbook of numerical analysis, Vol. VII*. Handb. Numer. Anal., VII. North-Holland, Amsterdam, 2000, pp. 713–1020.

[4] Bilal Saad and Mazen Saad. "Study of full implicit petroleum engineering finite-volume scheme for compressible two-phase flow in porous media". In: *SIAM Journal on Numerical Analysis* 51.1 (2013), pp. 716–741.

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- **Numerical scheme:** Backward Euler scheme in time and Two-Point Flux Approximation finite volume scheme[3][4] in space are employed.

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- **Numerical scheme:** Backward Euler scheme in time and Two-Point Flux Approximation finite volume scheme[3][4] in space are employed.

- **Energy estimates at the discrete level:**

$$\|p_{\mathscr{D}}\|_{L^2(0,t_F;H^1(\Omega))} + \|T_{\mathscr{D}}\|_{L^2(0,t_F;H^1(\Omega))} + \|\mathbf{u}_{\mathscr{D}}\|_{L^2(0,t_F;H^1_0(\Omega))^d} \leq C,$$

where $\mathscr{D}$ is the space-time discretization.

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[3] Robert Eymard, Thierry Gallouët, and Raphaèle Herbin. "Finite volume methods". In: *Handbook of numerical analysis, Vol. VII*. Handb. Numer. Anal., VII. North-Holland, Amsterdam, 2000, pp. 713–1020.

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Numerical results

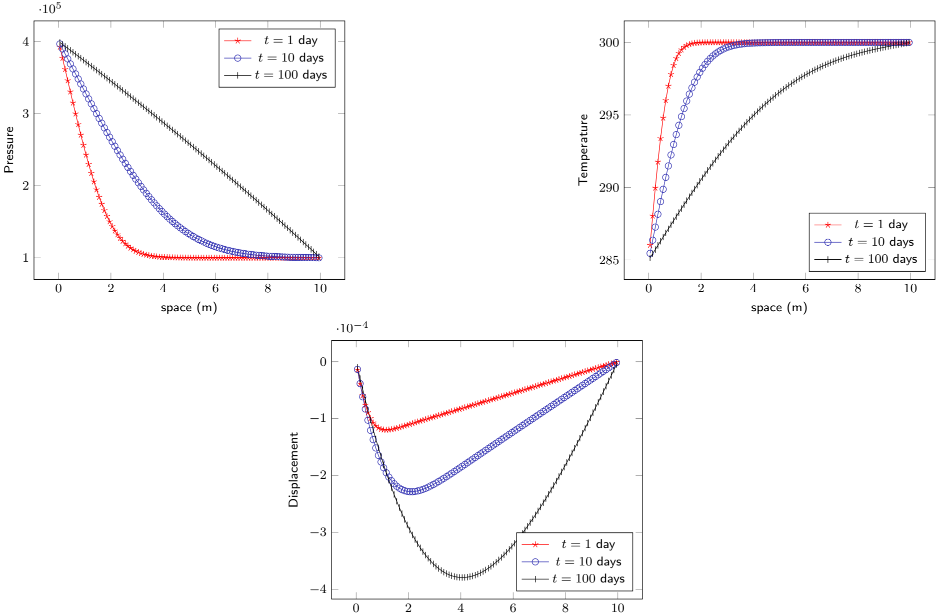
Numerical solution at different values of time t

We consider a porous medium of **length** 10 m, **initial porosity** $\phi_0 = 0.2$ and absolute **permeability** $\mathbb{K} = 10^{-15}\, m^2$.

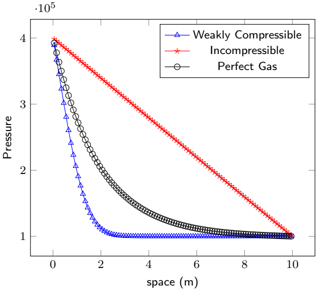
Moreover, we consider a **weakly compressible** fluid.

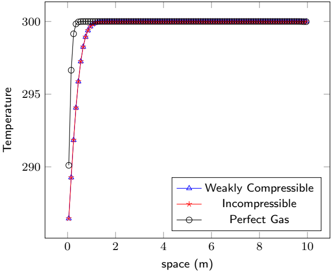
Initial Conditions: $p^0 = 10^5 Pa$, $T^0 = 300K$, and $u^0 = 10^{-5} m$.

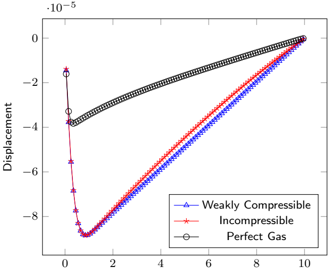
Boundary Conditions: The pressure: $p = 4 \times 10^5 Pa$ on the left and $p = 10^5 Pa$ on the right. The temperature: $T = 285K$ on the left and $T = 300K$ on the right. The displacement field satisfies homogeneous Dirichlet boundary conditions.


$\cdot 10^5$
Pressure
space (m)
$t = 1$ day
$t = 10$ days
$t = 100$ days
Temperature
space (m)
$t = 1$ day
$t = 10$ days
$t = 100$ days
$\cdot 10^{-4}$
Displacement
$t = 1$ day
$t = 10$ days
$t = 100$ days

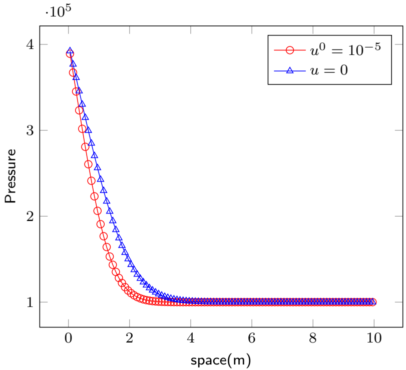
Influence of fluid compressibility

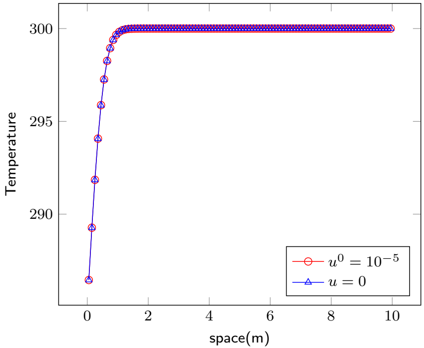




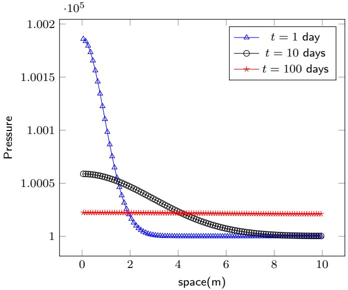


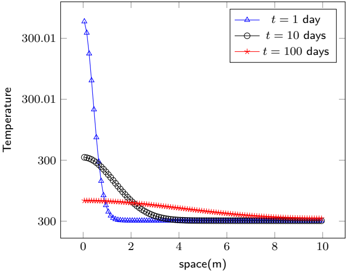
Influence of displacement: Small variations of displacement

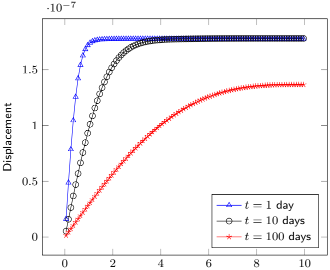




Neumann boundary conditions







Conclusion

Conclusion & Perspectives

Conclusion:

- We established energy estimates for the continuous formulation.
- We designed a numerical scheme that preserves the energy balance at the discrete level.
- We validated the model through numerical simulations demonstrating stability and physical consistency.

M.Mohamad, J.Dabaghi, F.Grondin, and M.Saad, *Finite Volume Discretization Preserving Energy Estimates for a Compressible THM Model*, submitted for publication, 2025.

Ongoing work:

- Extend the model to two-phase flow with one component in each phase.

M.Mohamad, J.Dabaghi, F.Grondin, and M.Saad, *Energy estimates and convergence analysis of a two-phase flow in deformable porous media*, submitted for publication, 2026.

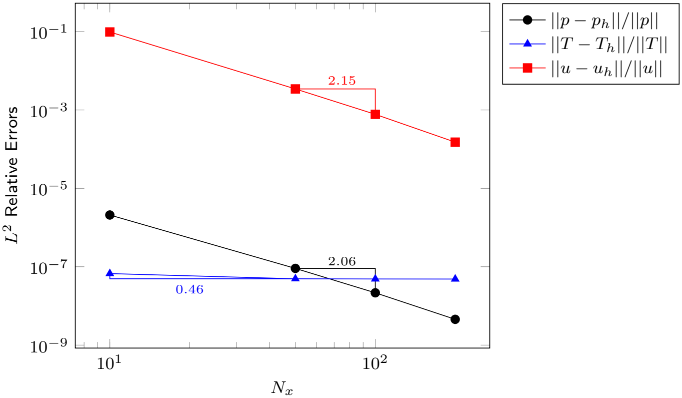
Future work:

- Investigate more complex configurations with two constituents in the gas phase and one component in the liquid phase.

Thank you for your attention!

The relative L^2 space-time error is defined, for example for the pressure, as

$$\frac{\|p - p_h\|_{L^2(\Omega, L^2(0,t_F))}}{\|p\|_{L^2(\Omega, L^2(0,t_F))}} = \frac{\left(\sum_{n=1}^{N_T} \sum_{K\in\mathcal{T}} (p(t^n, x_K) - p_h^n)^2\right)^{1/2}}{\left(\sum_{n=1}^{N_T} \sum_{K\in\mathcal{T}} p^2(t^n, x_K)\right)^{1/2}}.$$



Energy estimates for the compressible model: key identity

Lemma

$$\int_0^{t_F}\int_\Omega \partial_t E(\boldsymbol{u},p,T)\,\mathrm{d}x\mathrm{d}t + \int_0^{t_F}\int_\Omega \frac{m_0}{2}\partial_t(|\partial_t \boldsymbol{u}|^2)\,\mathrm{d}x\mathrm{d}t + \int_0^{t_F}\frac{\mathbb{K}}{\mu}\|\boldsymbol{\nabla} p\|^2_{L^2(\Omega)}(t)\mathrm{d}t$$
$$-\int_0^{t_F}\int_\Omega \phi(\boldsymbol{u},p,T)\frac{\rho'(p)}{\rho(p)}p\partial_t p\,\mathrm{d}x\mathrm{d}t + \int_0^{t_F}\frac{\lambda}{T_{\mathrm{ref}}}\|\boldsymbol{\nabla} T\|^2_{L^2(\Omega)}(t)\mathrm{d}t$$
$$+\int_0^{t_F}\int_\Omega \frac{\mathbb{K}(\rho(p))'}{\mu\rho(p)}p|\boldsymbol{\nabla} p|^2\,\mathrm{d}x\mathrm{d}t = \int_0^{t_F}\int_\Omega (h_m(g(p)+e-sT)+h_e+\boldsymbol{h}\cdot\partial_t\boldsymbol{u})\,\mathrm{d}x\mathrm{d}t,$$

where $E(\boldsymbol{u},p,T)$ is the total energy of the system.

Elements of the proof

•Step 1: Reformulate the entropy conservation equation.
Using the following relation between the fluid specific entropy s and internal energy e

$$\partial_t e = T\partial_t s - p\partial_t \left(\frac{1}{\rho}\right), \quad \boldsymbol{\nabla} e = T\boldsymbol{\nabla} s - p\boldsymbol{\nabla}\left(\frac{1}{\rho}\right).$$

The entropy conservation equation can be reformulated as

$$\partial_t S_s(\boldsymbol{u}, p, T) + \frac{\rho(p)\phi(\boldsymbol{u}, p, T)}{T}\partial_t e - \frac{p\phi(\boldsymbol{u}, p, T)}{T\rho(p)}\partial_t \rho(p) + \frac{\rho(p)\boldsymbol{V}(p)}{T}\cdot\boldsymbol{\nabla} e - \frac{p\boldsymbol{V}(p)}{T\rho(p)}\cdot\boldsymbol{\nabla}\rho(p) + \frac{1}{T_{\text{ref}}}\text{div}\boldsymbol{q}(T) = \frac{h_e}{T} - sh_m.$$

•Step 2: Multiply the mass conservation equation by $(g(p) + e)$ where $g(p)$ is the primitive of $1/\rho(p)$, the reformulated entropy conservation equation by T and the momentum balance equation by $\partial_t \boldsymbol{u}$, then integrate over space and time and sum the resulting equations.

•Step 3: Treat each term of the resulting equation from step 2, using appropriate mathematical tools, the closure equations and the boundary conditions.

Energy estimates: Control on the solution

Assumptions

- **(A1)** The porosity is always positive i.e. there exists $\phi_* \in \mathbb{R}_+^*$ such that $\phi(\boldsymbol{u}, p, T) \geq \phi_* > 0$.
- **(A2)** The energy functional is positive meaning that $e(p, T) \geq 0 \ \forall p, T \in \mathbb{R}_+$ and $e - Ts$ is sub-quadratic in the sense that $\lim_{(|p|,|T|)\to+\infty} \dfrac{e - Ts}{|p|^2 + |T|^2} = 0$.
- **(A3)** The source terms h_m, $\boldsymbol{h}$ and h_e are bounded.
- **(A4)** The thermo-poro-elastic coefficients satisfy $C_s > 0$, $\frac{1}{N} > 0$, $\alpha_\phi \geq 0$, $\mathcal{E} > 0$ and $\nu \in (0, \frac{1}{2})$.
- **(A5)** There exists a constant $m_* > 0$ such that $m_0 \geq m_* > 0$.
- **(A6)** The density $\rho(p)$ is a strictly increasing function and there exists a positive constant γ such that $\rho'(p) = \gamma\rho(p)$.

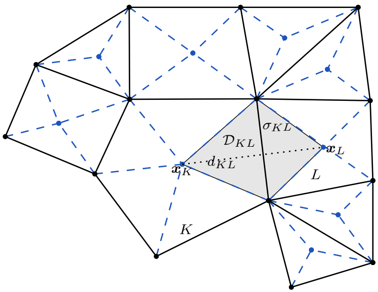
Theorem

We have the following energy estimate:

$$\|\partial_t \boldsymbol{u}\|_{L^\infty(0,t_F;L^2(\Omega))} + \|p\|_{L^\infty(0,t_F;L^2(\Omega))} + \|T\|_{L^\infty(0,t_F;L^2(\Omega))} + \|e\|_{L^\infty(0,t_F;L^1(\Omega))}$$
$$+ \|\boldsymbol{u}\|_{L^\infty(0,t_F;H^1(\Omega))} + \|\boldsymbol{\nabla} p\|_{L^2(0,t_F;L^2(\Omega))} + \|\boldsymbol{\nabla} T\|_{L^2(0,t_F;L^2(\Omega))} \leq C.$$

The finite volume method: TPFA scheme

Backward Euler scheme in time and **Two-Point Flux Approximation** finite volume scheme



Discrete unknowns:

$$\boldsymbol{\mathcal{U}}_h^n := (\boldsymbol{\mathcal{U}}_K^n)_{K\in\mathcal{T}}, \quad \boldsymbol{\mathcal{U}}_K^n := (p_K^n, T_K^n, \boldsymbol{u}_K^n) \in \mathbb{R}^{2+d}$$ One value per cell, N_x: number of cells.

Time discretisation: Consider $t^0 = 0 < t^1 < \cdots < t^{N_T} = t_F = N_T \delta t$ with constant time step δt. For $1 \le n \le N_T$, we define the backward differencing operator

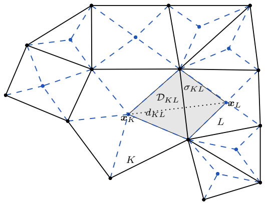
$$\delta_t^n f := \frac{f^n - f^{n-1}}{\delta t}.$$

At each time step, we solve a nonlinear problem involving $N_x \times (2+d)$ unknowns.

Space discretisation:

L_h is the discrete space of cellwise constant functions defined as

$$L_h := \left\{ v_h \in L^2(\Omega),\, v_h|_K = v_K \,\forall K \in \mathcal{T} \right\}.$$



For a piecewise constant function $v_h \in L_h$ defined per control volume, we define the discrete broken gradient $\nabla_h v_h \in \mathbb{R}^d$ by

$$\nabla_h v_h(x) := \begin{cases} d\frac{v_L - v_K}{d_{KL}} \boldsymbol{n}_{KL} & \text{if } \boldsymbol{x} \in \mathcal{D}_{KL}, \\ d\frac{v_\sigma - v_K}{d_{K\sigma}} \boldsymbol{n}_{K\sigma} & \text{if } \boldsymbol{x} \in \mathcal{D}_{K\sigma}. \end{cases}$$

For a piecewise constant vector function $\boldsymbol{w}_h \in L_h^d$ defined per control volume, we define the discrete broken gradient $\nabla_h \boldsymbol{w}_h \in \mathbb{R}^{d\times d}$ by

$$\nabla_h \boldsymbol{w}_h(x) := \begin{cases} d\frac{\boldsymbol{w}_L - \boldsymbol{w}_K}{d_{KL}} \otimes \boldsymbol{n}_{KL} & \text{if } \boldsymbol{x} \in \mathcal{D}_{KL}, \\ d\frac{\boldsymbol{w}_\sigma - \boldsymbol{w}_K}{d_{K\sigma}} \otimes \boldsymbol{n}_{K\sigma} & \text{if } \boldsymbol{x} \in \mathcal{D}_{K\sigma}. \end{cases}$$

We define the discrete divergence of the vector field $\boldsymbol{G}_h$ by

$$\mathrm{div}_K \boldsymbol{G}_h := \frac{1}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{KL}| \boldsymbol{G}_{KL} \cdot \boldsymbol{n}_{KL}.$$

The finite volume scheme: Mass conservation equation

Mass conservation equation:

$$\int_{t^{n-1}}^{t^n}\int_K \partial_t(\rho(p)\phi(\boldsymbol{u},p,T))\,\mathrm{d}x\mathrm{d}t - \frac{k}{\mu}\int_{t^{n-1}}^{t^n}\int_{\partial K}\rho(p)\boldsymbol{\nabla}(p)\cdot\mathbf{n}\,d\sigma dt = \int_{t^{n-1}}^{t^n}\int_K h_m(x,t)\,dxdt,$$

which reads

$$|K|\left(\rho_K^n\phi_K^n - \rho_K^{n-1}\phi_K^{n-1}\right) + \delta t \sum_{L\in\mathcal{N}(K)} \mathcal{F}_{1,KL}(\boldsymbol{\mathcal{U}}^n) = \delta t|K|h_{m,K}^n,$$

with $\mathcal{F}_{1,KL}^n(\boldsymbol{\mathcal{U}}^n) := -\rho_{KL}^n\frac{k}{\mu}\tau_{KL}(p_L^n - p_K^n).$

where ρ_{KL}^n is the value of the density on the interface σ_{KL} and it is defined by,

$$\frac{1}{\rho_{KL}^n} := \begin{cases} \dfrac{1}{p_K^n - p_L^n}\displaystyle\int_{p_L^n}^{p_K^n}\frac{1}{\rho(z)}dz & \text{if} \quad p_K^n \neq p_L^n, \\ \frac{1}{\rho_K^n} & \text{otherwise.}\end{cases}$$

The finite volume scheme: Entropy conservation equation

Entropy conservation equation:

$$\int_{t^{n-1}}^{t^n}\int_K \partial_t(S_s(\boldsymbol{u},p,T)+\rho(p)\phi(\boldsymbol{u},p,T)s)\,\mathrm{d}x\mathrm{d}t$$
$$-\int_{t^{n-1}}^{t^n}\int_{\partial K}\left(\frac{k}{\mu}\rho(p)s(T)\boldsymbol{\nabla}p+\frac{\lambda}{T_{\mathrm{ref}}}\boldsymbol{\nabla}T\right)\cdot\boldsymbol{n}\,d\sigma\mathrm{d}t=\int_{t^{n-1}}^{t^n}\int_K\frac{h_e(x,t)}{T}\,\mathrm{d}x\mathrm{d}t,$$

which reads

$$|K|\left(S^n_{s,K}+\rho^n_K\phi^n_K s^n_K-S^{n-1}_{s,K}-\rho^{n-1}_K\phi^{n-1}_K s^{n-1}_K\right)+\delta t\sum_{L\in\mathcal{N}(K)}\mathcal{F}^n_{2,KL}(\boldsymbol{\mathcal{U}}^n)=\delta t|K|\frac{h^n_{e,K}}{T^n_K},$$

with $\mathcal{F}^n_{2,KL}(\boldsymbol{\mathcal{U}}^n):=-\rho^n_{KL}s^n_{KL}\frac{k}{\mu}\tau_{KL}(p^n_L-p^n_K)-\frac{\lambda}{T_{\mathrm{ref}}}\tau_{KL}\left(T^n_L-T^n_K\right)$

where

$$s^n_{KL}:=\frac{s^n_K+s^n_L}{2},$$

is the value of the fluid entropy s on the interface σ_{KL}.

The finite volume scheme: Momentum balance equation

Momentum balance equation:

$$\int_{t^{n-1}}^{t^n}\int_K m_0 \partial_{tt}^2 \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t - \int_{t^{n-1}}^{t^n}\int_{\partial K} \boldsymbol{\sigma}(\boldsymbol{u},p,T)\boldsymbol{n}\,d\sigma\mathrm{d}t = \int_{t^{n-1}}^{t^n}\int_K \boldsymbol{h}(x,t)\,\mathrm{d}x\mathrm{d}t,$$

which reads

$$|K|\frac{\boldsymbol{u}_K^n + \boldsymbol{u}_K^{n-2} - 2\boldsymbol{u}_K^{n-1}}{\delta t} + \delta t \sum_{L\in\mathcal{N}(K)} \mathcal{F}_{3,KL}^n(\boldsymbol{\mathcal{U}}^n) = \delta t|K|\boldsymbol{h}_K^n, \text{ with}$$

$$\begin{aligned}\mathcal{F}_{3,KL}(\boldsymbol{\mathcal{U}}^n) :=& \frac{-\mathcal{E}}{(1+\nu)}\frac{\tau_{KL}}{2}\left[((\boldsymbol{u}_L-\boldsymbol{u}_K)\otimes\boldsymbol{n}_{KL})\,\boldsymbol{n}_{KL} + [(\boldsymbol{u}_L-\boldsymbol{u}_K)\otimes\boldsymbol{n}_{KL}]^\top\boldsymbol{n}_{KL}\right]\\ &- \frac{\mathcal{E}\nu}{(1+\nu)(1-2\nu)}|\sigma_{KL}|\,(\mathrm{div}\boldsymbol{u}^n\mathbb{I}_d)_{KL}\,\boldsymbol{n}_{KL} + b|\sigma_{KL}|p_{KL}^n\mathbb{I}_d\boldsymbol{n}_{KL}\\ &+ |\sigma_{KL}|3\alpha_s K_s\,(T_{KL}^n - T_{\mathrm{ref}})\,\mathbb{I}_d\boldsymbol{n}_{KL}.\end{aligned}$$

where

$$(\mathrm{div}\boldsymbol{u}^n\mathbb{I}_d)_{KL} := \frac{\mathrm{div}_K\boldsymbol{u}_h^n + \mathrm{div}_L\boldsymbol{u}_h^n}{2}\mathbb{I}_d, \quad p_{KL}^n := \frac{p_K^n + p_L^n}{2}, \quad T_{KL}^n := \frac{T_K^n + T_L^n}{2}.$$

are the values of the discrete divergence of $\boldsymbol{u}$, fluid pressure p and temperature T on the interface σ_{KL}, respectively.

Discrete variational formulation

Proposition

The finite volume scheme is equivalent to the following **discrete variational formulation**:

$$\int_\Omega \frac{\rho_h^n \phi_h^n - \rho_h^{n-1}\phi_h^{n-1}}{\delta t}\varphi_h \,\mathrm{d}x + \frac{1}{d}\int_\Omega \rho_h^n \frac{k}{\mu}\boldsymbol{\nabla}_h p_h^n \cdot \boldsymbol{\nabla}_h \varphi_h \,\mathrm{d}x = \int_\Omega h_{m,h}^n \varphi_h \,\mathrm{d}x. \quad (1)$$

$$\begin{aligned}&\int_\Omega \frac{S_{s,h}^n - S_{s,h}^{n-1}}{\delta t}\psi_h \,\mathrm{d}x + \int_\Omega \frac{\rho_h^{n-1}\phi_h^{n-1}}{T_h^n}\left(\delta_t^n e_h + p_h^n \delta_t^n\left(\frac{1}{\rho_h}\right)\right)\psi_h \,\mathrm{d}x \\ &- \frac{1}{d}\int_\Omega \frac{\rho_h^n k}{T_h^n \mu}\boldsymbol{\nabla}_h p_h^n \cdot \boldsymbol{\nabla}_h e_h^n \psi_h \,\mathrm{d}x - \frac{1}{d}\int_\Omega \frac{\rho_h^n p_h^n}{T_h^n}\frac{k}{\mu}\boldsymbol{\nabla}_h\left(\frac{1}{\rho_h^n}\right)\cdot \boldsymbol{\nabla}_h p_h^n \psi_h \,\mathrm{d}x \\ &+ \frac{1}{d}\int_\Omega \frac{\lambda}{T_{\mathrm{ref}}}\boldsymbol{\nabla}_h T_h^n \cdot \boldsymbol{\nabla}_h \psi_h \,\mathrm{d}x = \int_\Omega \left(\frac{h_{e,h}^n}{T_h^n} + s_h^n h_{m,h}^n\right)\psi_h \,\mathrm{d}x.\end{aligned} \quad (2)$$

$$\begin{aligned}&\int_\Omega \delta_{tt}^n \boldsymbol{u}_h \cdot \boldsymbol{\omega}_h \,\mathrm{d}x + \frac{1}{d}\int_\Omega \frac{\mathcal{E}}{(1+\nu)}\epsilon(\boldsymbol{u}_h^n) : \epsilon(\boldsymbol{\omega}_h)\,\mathrm{d}x + \int_\Omega \frac{\mathcal{E}\nu}{(1+\nu)(1-2\nu)}\mathrm{div}_h \boldsymbol{u}_h^n \mathbb{I}_d : \epsilon(\boldsymbol{\omega}_h)\,\mathrm{d}x \\ &- \int_\Omega (b p_h^n + 3\alpha_s K_s (T_h^n - T_{\mathrm{ref}}))\,\mathrm{div}_h(\boldsymbol{\omega}_h)\,\mathrm{d}x = \int_\Omega \boldsymbol{h}_h^n \cdot \boldsymbol{\omega}_h \,\mathrm{d}x.\end{aligned} \quad (3)$$

$\forall(\varphi_h, \psi_h, \boldsymbol{\omega}_h) \in H_h^{2+d}$.

Discrete energy estimates: key inequality

Lemma

Let $(p_h^n, T_h^n, \boldsymbol{u}_h^n)$ be the solution of the discrete variational formulation. We have the following inequality:

$$
\begin{aligned}
&\int_\Omega \delta_t^n E_h \,\mathrm{d}x + \int_\Omega \frac{m_0}{2}\delta_t^n |\delta_t^n \boldsymbol{u}_h|^2 \,\mathrm{d}x - \int_\Omega \phi_h^{n-1}\frac{\tilde{\rho}_h'}{\rho_h^n} p_h^n \delta_t^n p_h \,\mathrm{d}x + \frac{k}{\mu\, d}\|\boldsymbol{\nabla}_h p_h^n\|_{L^2(\Omega)}^2 \\
&+ \frac{\lambda}{T_{\mathrm{ref}}\, d}\|\boldsymbol{\nabla}_h T_h^n\|_{L^2(\Omega)}^2 + \frac{1}{d}\int_\Omega \overline{\rho}_h' \frac{k p_h^n}{\mu \rho_h^n}\left|\boldsymbol{\nabla}_h p_h^n\right|^2 \,\mathrm{d}x \\
&\leq \int_\Omega \Big(h_{m,h}^n (g(p_h^n) + e_h^n - s_h^n T_h^n) + h_{e,h}^n + \boldsymbol{h}_h^n \cdot \delta_t^n \boldsymbol{u}_h^n \Big) \,\mathrm{d}x.
\end{aligned}
$$

Discrete energy estimates: Control on the solution

Let $\mathscr{D}$ be a discretization of $\Omega \times (0, t_F)$. We denote any function from $\mathcal{T} \times [0, N_T + 1]$ to $\mathbb{R}$ by using the subscript $\mathscr{D}$, and we denote its value at the point (x_K, t^n) using the subscript K and the superscript n, we then denote $p_{\mathscr{D}} = (p_K^n)_{K \in \mathcal{T}, n \in [0, N_T+1]}$.

Proposition

Let $(p_{\mathscr{D}}, T_{\mathscr{D}}, \mathbf{u}_{\mathscr{D}})$ be the solution of the discrete variational formulation. We assume that the previous assumptions hold for the discrete parameters. We obtain the following estimate:

$$\|\partial_t \boldsymbol{u}_{\mathscr{D}}\|_{L^\infty(0,t_F;L^2(\Omega))} + \|p_{\mathscr{D}}\|_{L^\infty(0,t_F;L^2(\Omega))} + \|T_{\mathscr{D}}\|_{L^\infty(0,t_F;L^2(\Omega))} + \|e_{\mathscr{D}}\|_{L^\infty(0,t_F;L^1(\Omega))}$$
$$+ \|\boldsymbol{u}_{\mathscr{D}}\|_{L^\infty(0,t_F;H^1(\Omega))} + \|\boldsymbol{\nabla}_h p_{\mathscr{D}}\|_{L^2(0,t_F;L^2(\Omega))} + \|\boldsymbol{\nabla}_h T_{\mathscr{D}}\|_{L^2(0,t_F;L^2(\Omega))} \leq C.$$

Two-Phase flow

We consider two phases: one wetting denoted by w containing water and the other non-wetting denoted by nw containing hydrogen.

Wetting fluid mass conservation equation:

$$\partial_t(s_w \rho_w \phi) + \operatorname{div}(\rho_w \boldsymbol{V}_w(p_w)) = h_w.$$

Non-wetting fluid mass conservation equation:

$$\partial_t(s_{nw} \rho_{nw} \phi) + \operatorname{div}(\rho_{nw} \boldsymbol{V}_{nw}(p_{nw})) = h_{nw}.$$

Entropy conservation equation:

$$\partial_t \left(S_s + \sum_{\alpha \in \{w,nw\}} \rho_\alpha s_\alpha c_\alpha \phi \right) + \operatorname{div} \left(\sum_{\alpha \in \{w,nw\}} \rho_\alpha c_\alpha \boldsymbol{V}_\alpha(p_\alpha) + \frac{1}{T_{\text{ref}}} \boldsymbol{q}(T) \right) = \frac{h_e}{T}.$$

Momentum balance equation:

$$m_0 \partial_{tt}^2 \boldsymbol{u} - \operatorname{div} \boldsymbol{\sigma} = \boldsymbol{h}.$$

The main unknowns are the wetting fluid pressure (p_w), the non-wetting fluid pressure (p_{nw}), the fluid temperature (T) and the displacement of the skeleton $(\boldsymbol{u})$.

Darcy law:

$$\boldsymbol{V}_\alpha := -\mathbb{K}\frac{K_{r\alpha}(s_\alpha)}{\mu_\alpha}\boldsymbol{\nabla} p_\alpha.$$

Saturation:

$$s_w + s_{nw} = 1.$$

Capillary pressure:

$$p_c(s_w) = p_{nw} - p_w.$$

Linear isotropic thermo-poro-elastic constitutive laws:

$$\boldsymbol{\sigma}(\boldsymbol{u}, p_w, p_{nw}, T) := \boldsymbol{\sigma}^e(\boldsymbol{u}) - b\pi\mathbb{I}_d - 3\alpha_s K_s(T - T_{\mathrm{ref}})\mathbb{I}_d,$$

$$\partial_t\phi(\boldsymbol{u}, p_w, p_{nw}, T) = b\partial_t(\mathrm{div}\boldsymbol{u}) - 3\alpha_\phi\partial_t T + \frac{1}{N}\partial_t\pi,$$

$$\partial_t S_s(\boldsymbol{u}, p_w, p_{nw}, T) = 3\alpha_s K_s\partial_t(\mathrm{div}\boldsymbol{u}) - 3\alpha_\phi\partial_t\pi + \frac{C_s}{T_{\mathrm{ref}}}\partial_t T.$$

π denotes the **equivalent pore pressure:**

$$\pi := p^\star - U,$$

where $p^\star$ is **the averaged fluid pressure** defined by:

$$p^\star := s_w p_w + s_{nw} p_{nw},$$

and U is the **interfacial energy**, which is a function of saturation s_w and it is defined by:

$$U(s_w) := \int_{s_w}^{1} p_c(z)\,dz.$$