Two-Phase Flow of Yield Stress Fluids in Porous Media : Invasion patterns and Flow Regimes

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FAST

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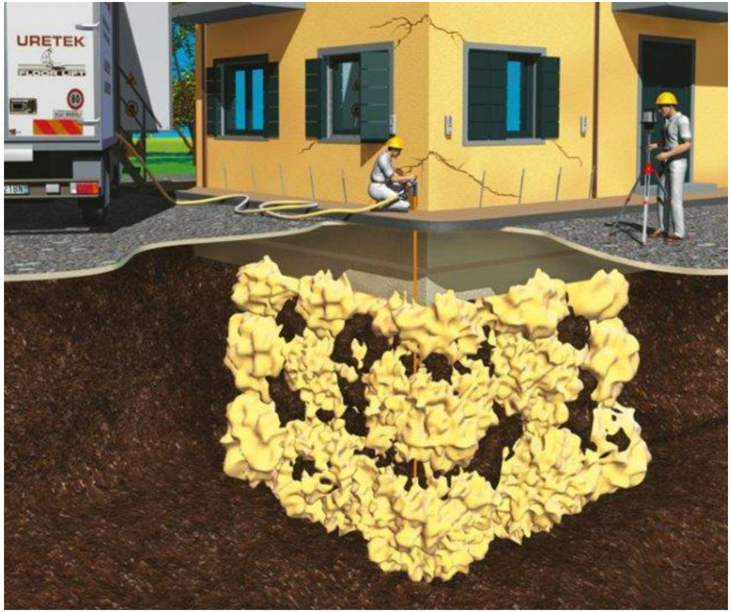
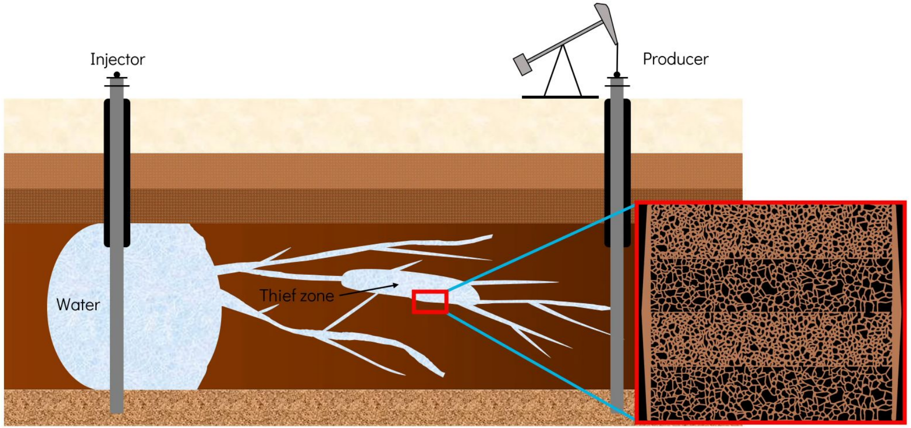
Flow of yield-stress fluid can be found in a range of industrial applications

Injection of foam



Comblement de cavités, **solscope.fr**

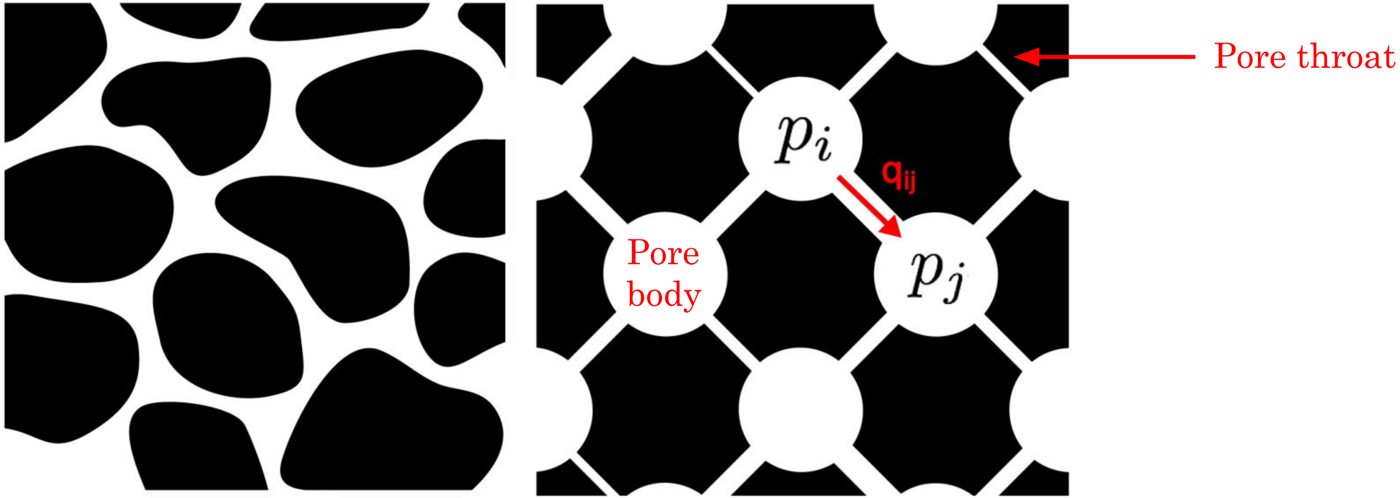
Enhanced Oil Recovery



interfacefluidics.com

Pore network model : a simple representation of a porous medium

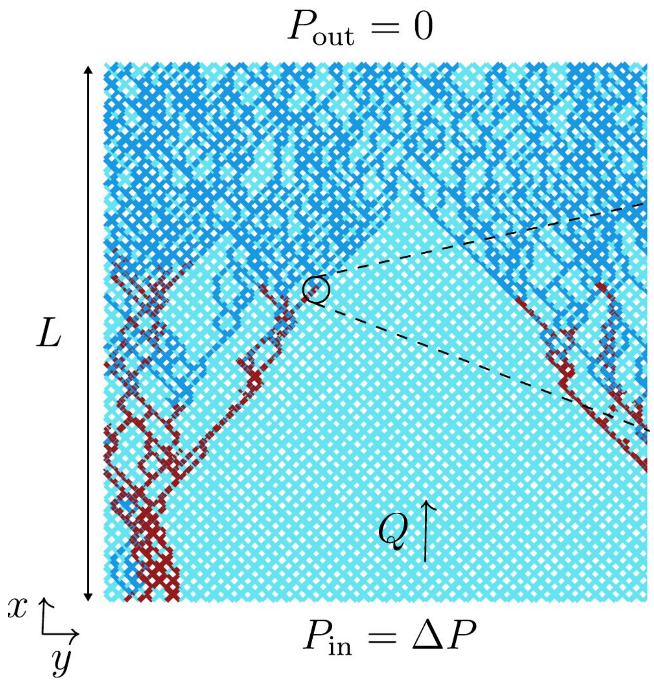
Heterogeneous network of cylindrical tubes filled with a yield-stress fluid



Disorder : Radii follow Uniform distribution

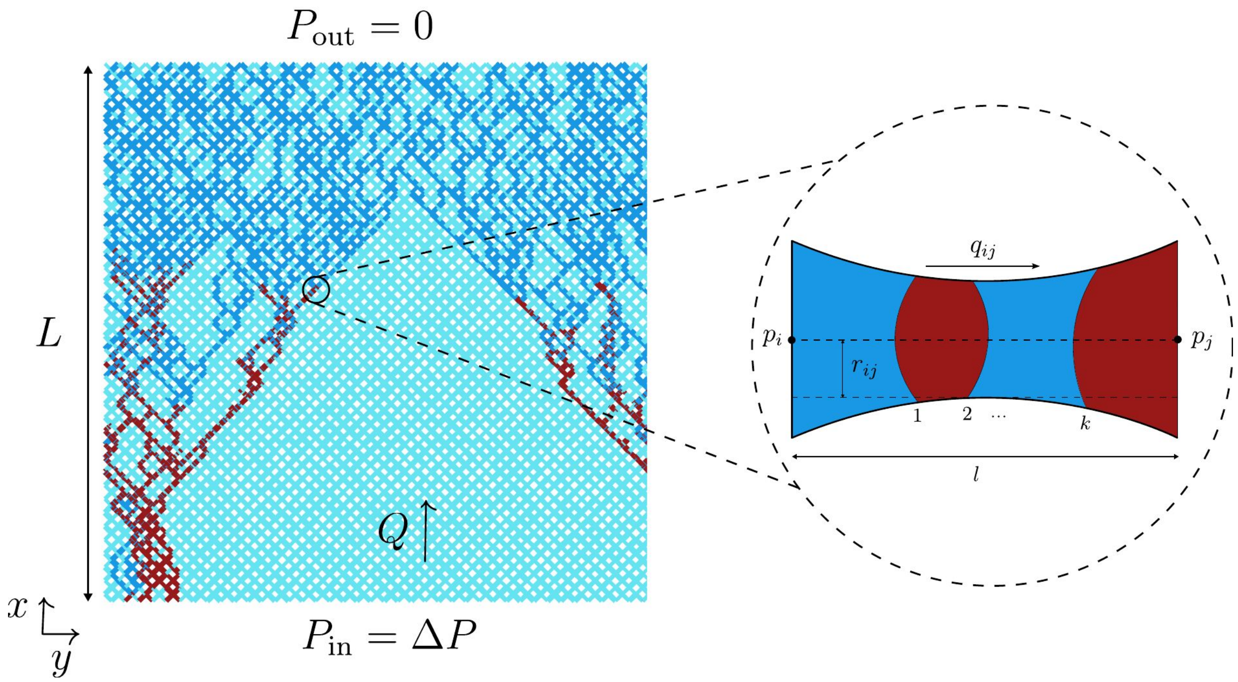
Mass conservation $$\sum_{j \in V(i)} q_{ij} = 0$$

Pore network model : a simple representation of a porous medium



- **Pressure** is imposed
- Newtonian fluid is injected at the bottom boundary
- Pressure is computed at each node
- Simulation stops once the Newtonian fluid reaches brerakthrough

Pore network model : a simple representation of a porous medium



In each link :

To flow, need to overcome

- The local pressure threshold :

$$\tau_{ij} = \frac{2\tau_c}{r_{ij}}$$

- The capillary barrier :

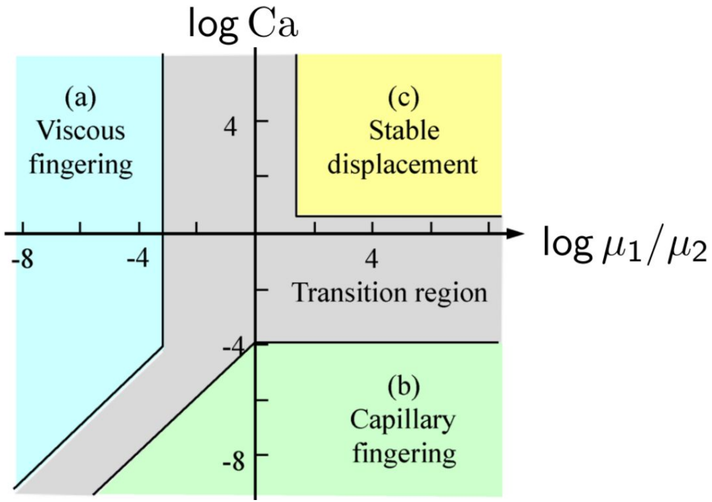
$$P_{ij}^{\sigma} = \sum_{k=1}^{n} \pm \frac{2\sigma}{r_{ij}} \left[1 - a \cos\left(\frac{2\pi x_k}{l} \right) \right]$$

There are multiple physical phenomena in competition with each other

From the Newtonian diagram :

- Viscosity ratio $M = \frac{\mu}{\eta}$ (invading / defending)
- **Viscous** forces vs **Capillary** forces

New parameter : **Yield-stress**



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- Viscosity ratio $M = \frac{\mu}{\eta}$ (invading / defending)
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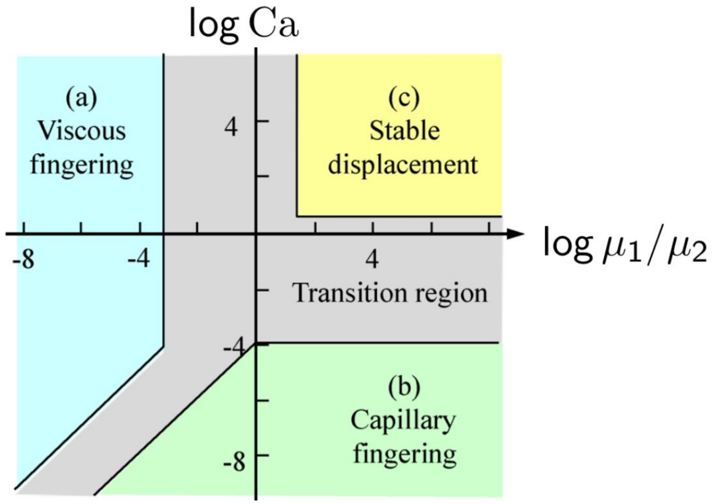
New parameter : **Yield-stress**

Dimensionless numbers :

- Bingham number $B_n = \frac{L\langle\tau\rangle}{\Delta P} = \frac{2\tau_c\langle 1/r\rangle L}{\Delta P}$
- Capillary number $Ca_P = \frac{\Delta P - \Delta P_0}{2\sigma/r_0}$
- “Ratio” number $R_\tau = \frac{L\langle\tau\rangle}{2\sigma/r_0} = \frac{L\tau_c}{\sigma}$



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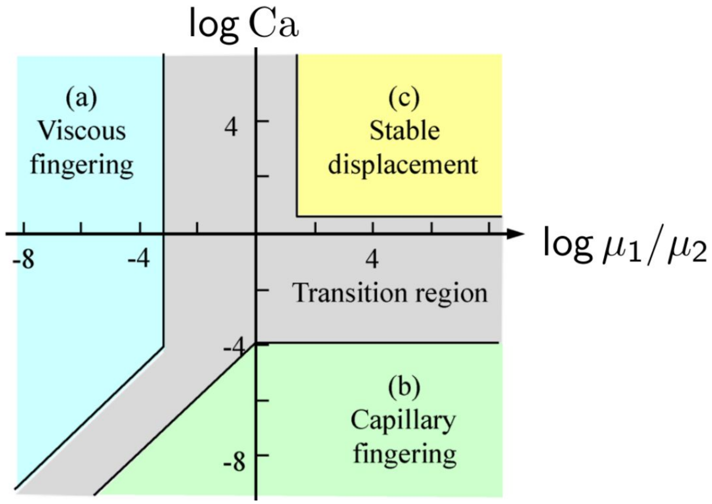
New parameter : **Yield-stress**

<u>**Our Goal**</u> : **Extend Lenormand's diagram**



Hierarchy of preferential flowing paths are due to yield stress and disorder

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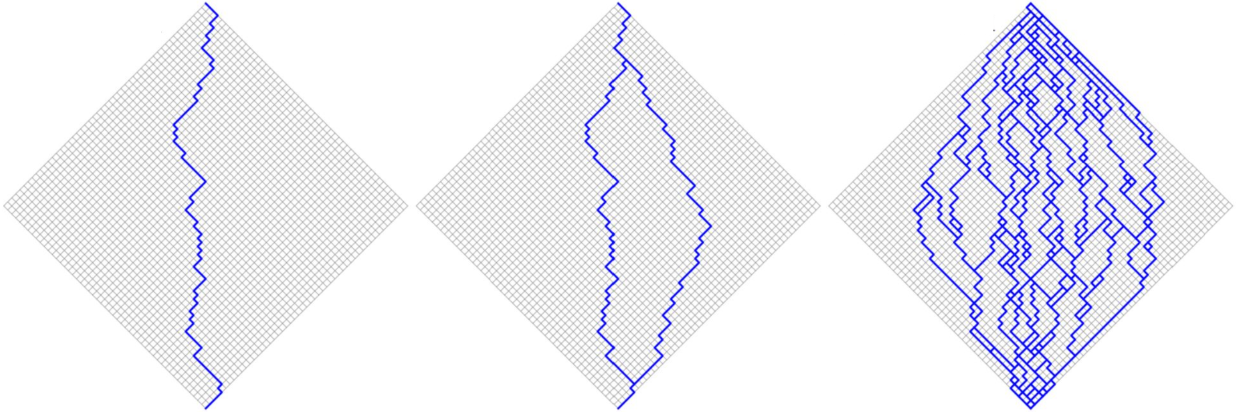


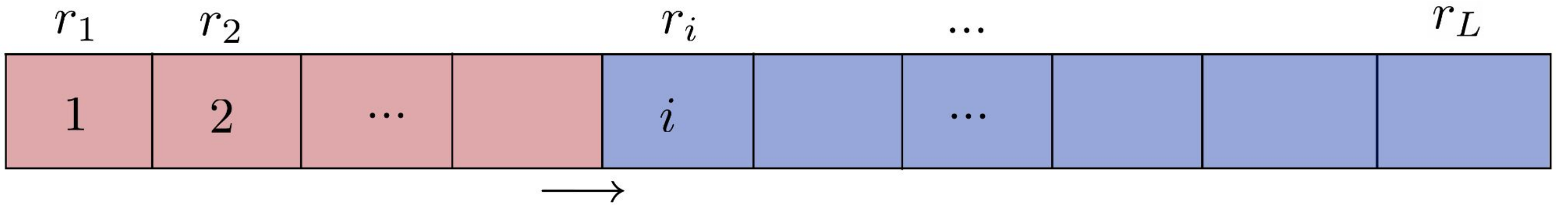
Liu et al., PRL 2019

$\Delta P_0 = \min_{\mathcal{C}} \sum_{\langle ij \rangle \in \mathcal{C}} \tau_{ij}$ $\quad$ $\Delta P_1 < \Delta P < \Delta P_2$ $\quad$ $\Delta P_k < \Delta P < \Delta P_{k+1}$

In the single phase case, each path must overcome yield condition to flow

A breakthrough criterion can be derived in 1D

And it’s not so trivial!



$$c_i = \frac{4a\sigma}{r_i} + \sum_{j=i}^{N} \frac{2\tau_c}{r_j} \Longrightarrow \Delta P_0 = \max_{i=1,\ldots,L} c_i$$

Pressure threshold : Maximum of constraint values

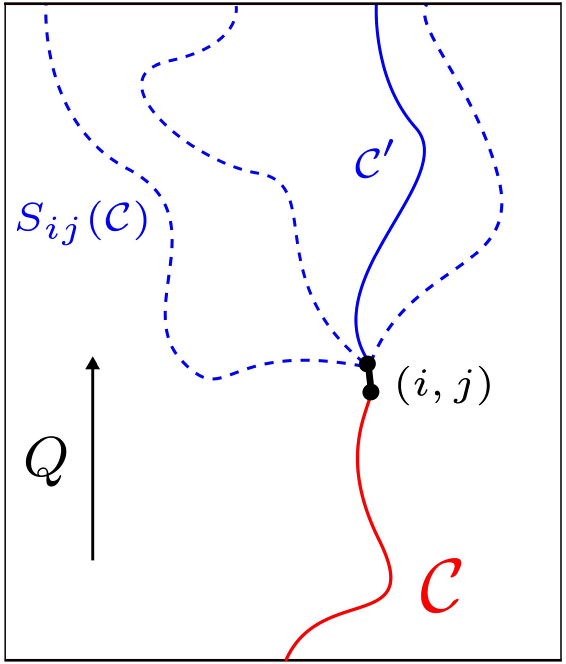
$\Longrightarrow$ Extreme values of strongly correlated random variables

A breakthrough criterion can be derived in 2D too

For a candidate path $\mathcal{C}$, the constraint to overcome for a given link $\langle ij \rangle \in \mathcal{C}$ is

$$c_{ij} = \underbrace{\frac{4\sigma a}{r_{ij}}}_{\text{Capillary term}} + \underbrace{\min_{\mathcal{C}' \in \mathcal{S}_{ij}(\mathcal{C})} \sum_{\langle kl \rangle \in \mathcal{C}'} \tau_{kl}}_{\text{Yield stress term}}$$

$\mathcal{S}_{ij}(\mathcal{C})$: set of paths starting from node j that avoid any link in $\mathcal{C}$ up to link $\langle ij \rangle \in \mathcal{C}$



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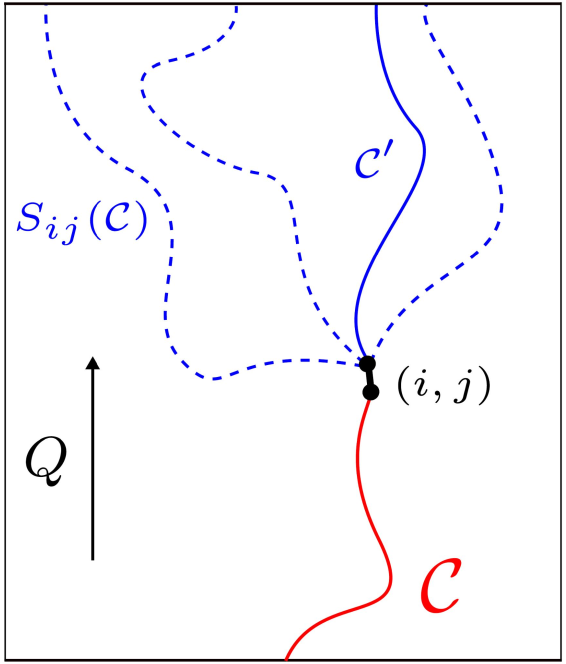
$\mathcal{S}_{ij}(\mathcal{C})$: set of paths starting from node j that avoid any link in $\mathcal{C}$ up to link $\langle ij \rangle \in \mathcal{C}$

To reach breakthrough with this path,

$$\Delta P > \max_{\langle ij \rangle \in \mathcal{C}} c_{ij}$$



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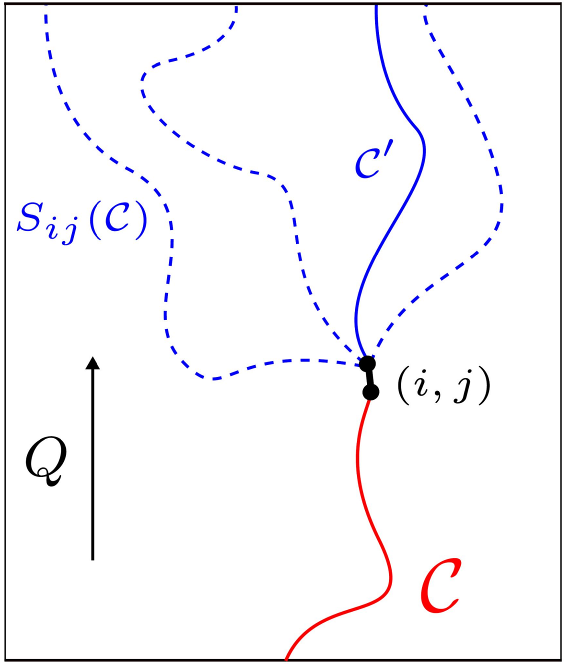
Considering all possible paths, we get

the critical pressure threshold :



$$\Delta P_0 = \min_{\mathcal{C}} \max_{\langle ij \rangle \in \mathcal{C}} \left[\frac{4a\sigma}{r_{ij}} + \min_{\mathcal{C}' \in \mathcal{S}_{ij}(\mathcal{C})} \sum_{\langle kl \rangle \in \mathcal{C}'} \tau_{kl} \right]$$

The breakthrough line

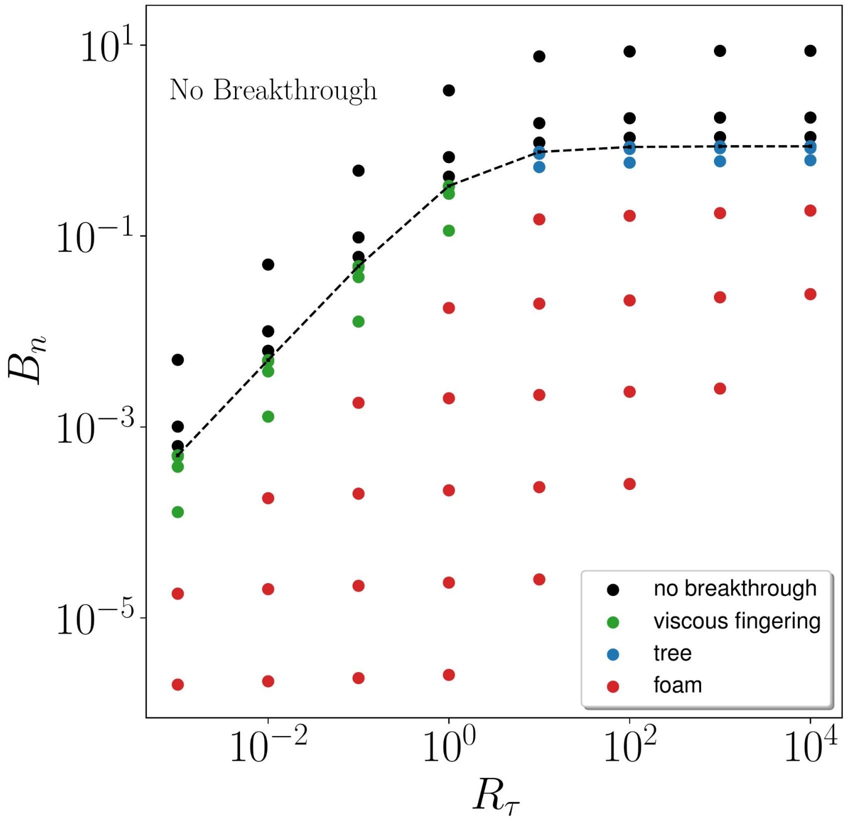
Breakthrough condition : $\Delta P > \Delta P_0$

with :

$$\Delta P_0 = \min_{\mathcal{C}} \max_{\langle ij \rangle \in \mathcal{C}} \left[\frac{4a\sigma}{r_{ij}} + \min_{\mathcal{C}' \in \mathcal{S}_{ij}(\mathcal{C})} \sum_{\langle kl \rangle \in \mathcal{C}'} \tau_{kl} \right]$$

and the Bingham number $B_n = \dfrac{L\langle \tau \rangle}{\Delta P}$

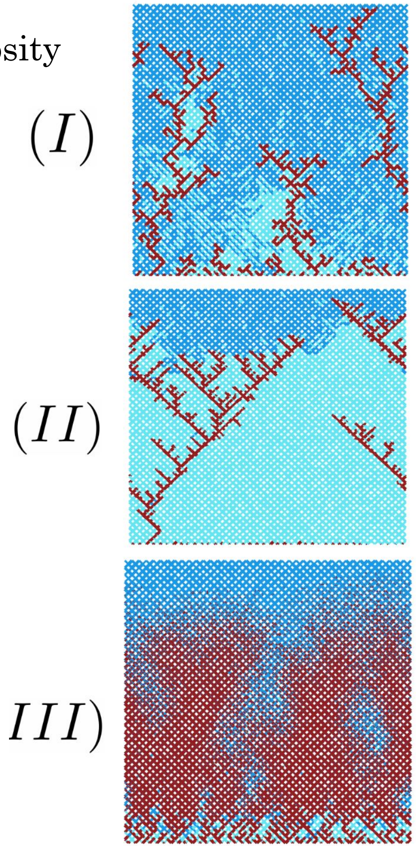
$M = 10^{-3}$

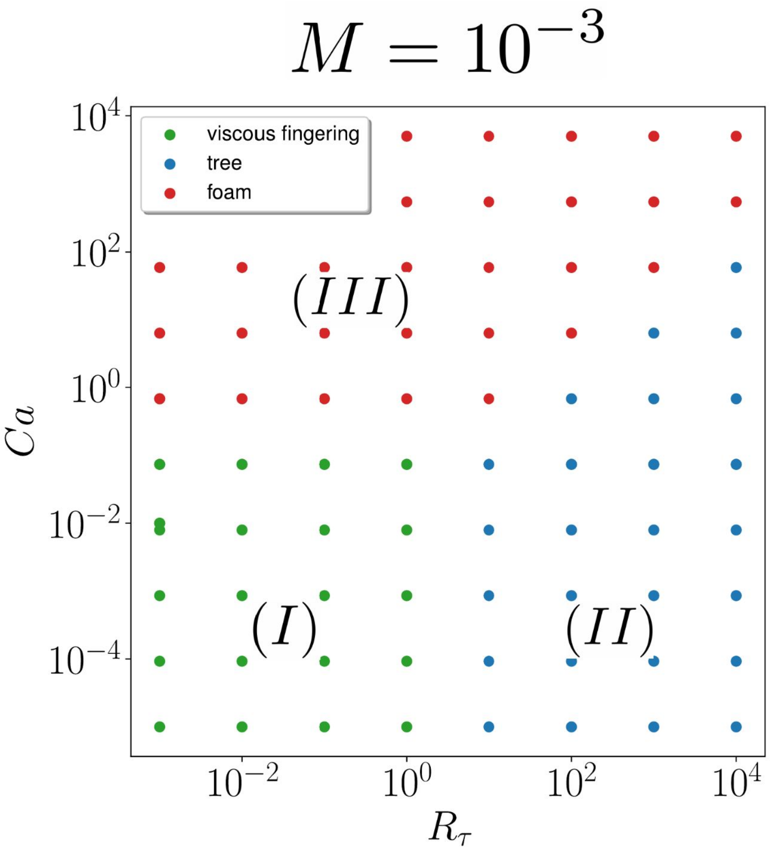


The Phase diagram

M = Newtonian viscosity / YSF viscosity

$$R_\tau = \frac{L\tau_c}{\sigma}$$





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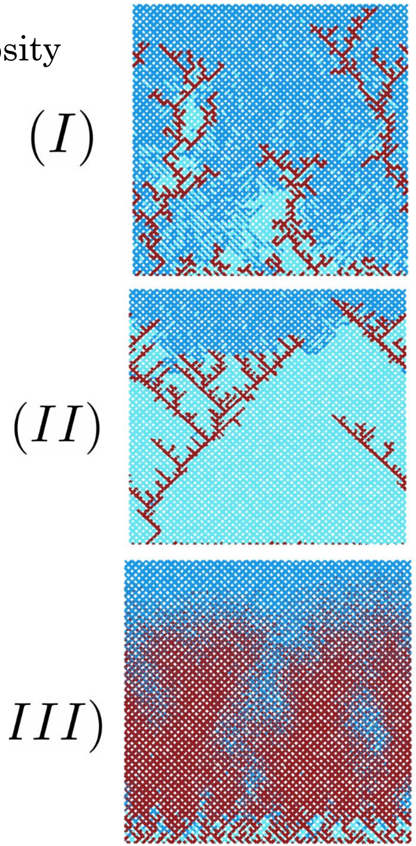

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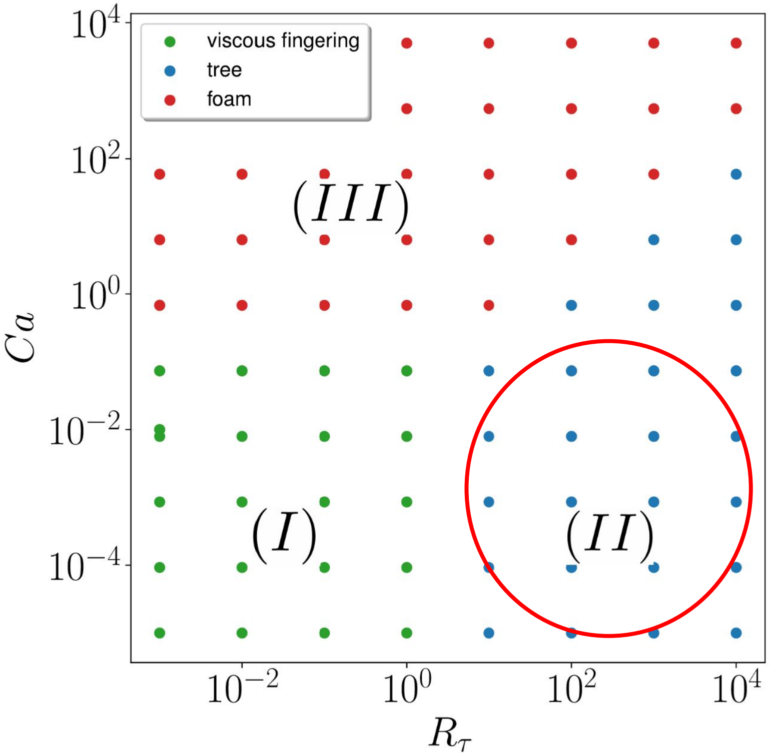
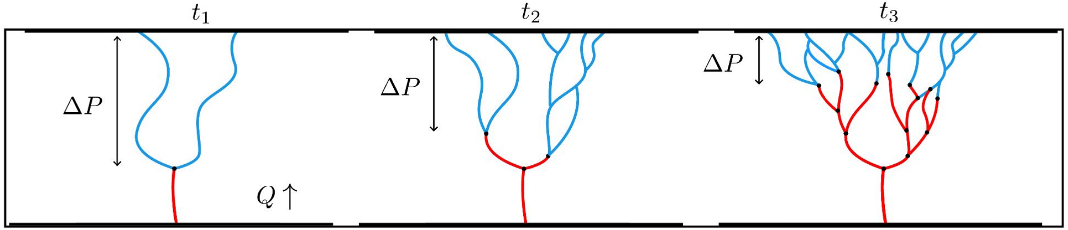
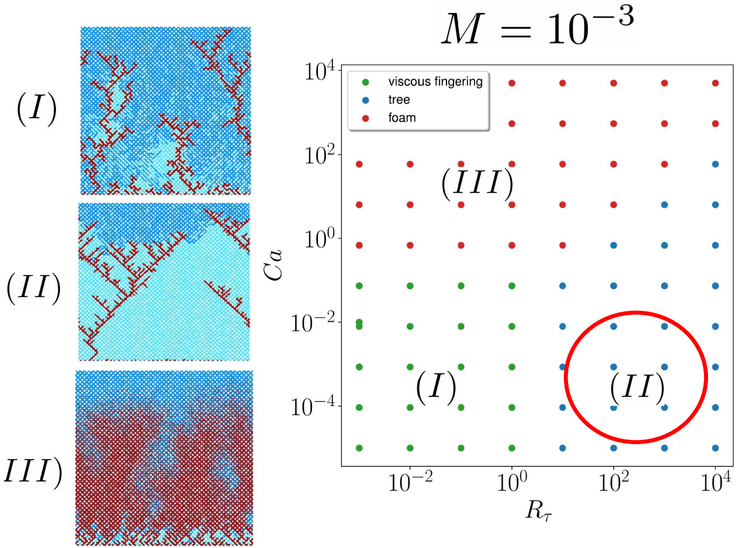


The Directed Tree regime

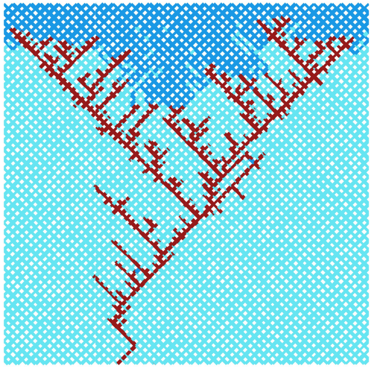
M = Newtonian viscosity / YSF viscosity

$$R_\tau = \frac{L\tau_c}{\sigma}$$





Average tortuosity:



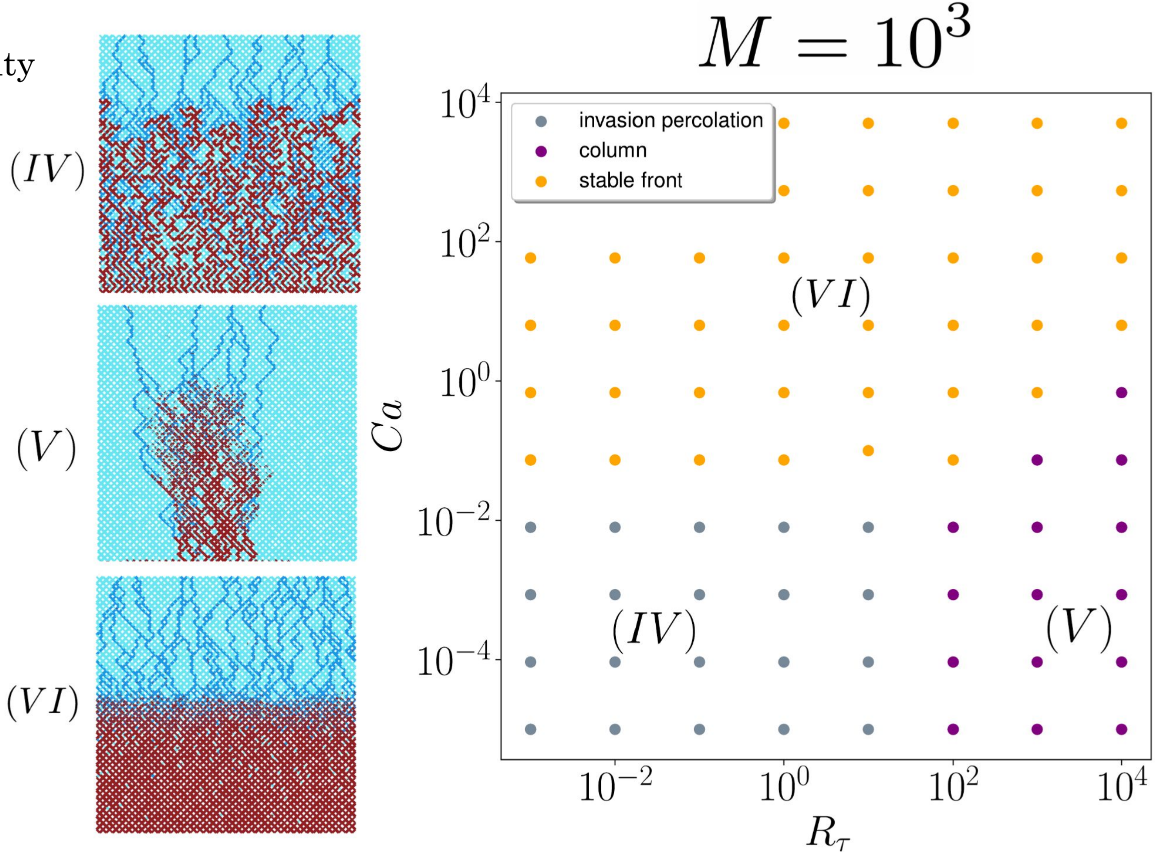
Flow rate - saturation scaling law:

$$Q = n_{\text{tips}} Q_{\text{tip}} \sim {S_n}^{2/D}$$

The Phase diagram

M = Newtonian viscosity / YSF viscosity

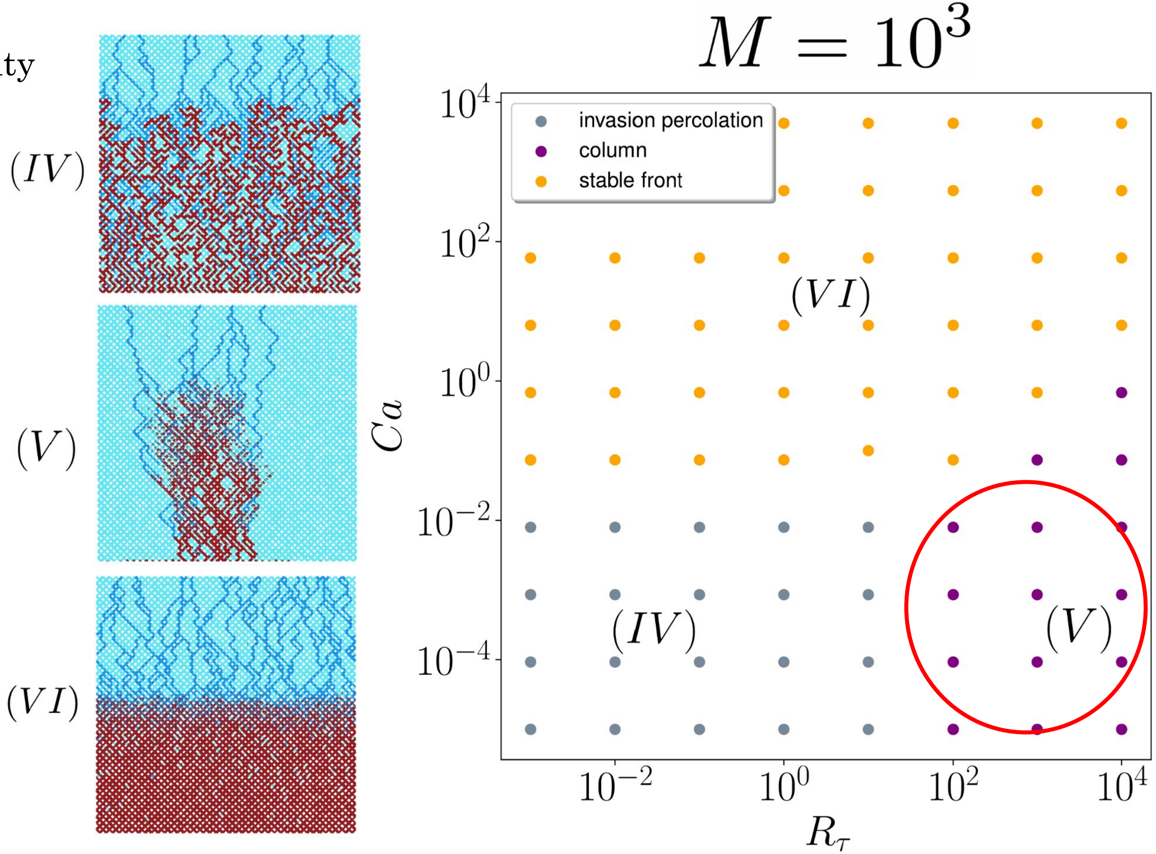
$$R_\tau = \frac{L\tau_c}{\sigma}$$



The Phase diagram

M = Newtonian viscosity / YSF viscosity

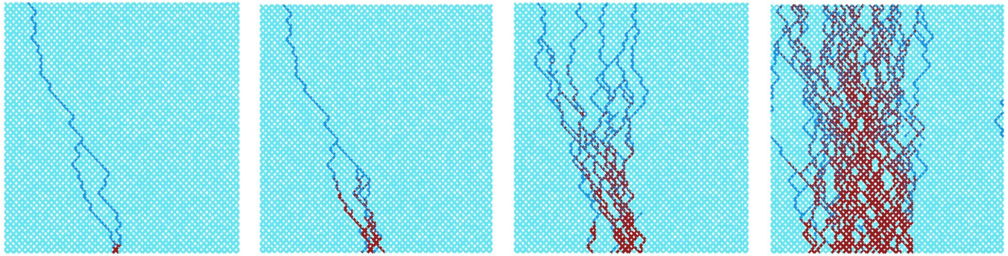
$$R_\tau = \frac{L\tau_c}{\sigma}$$

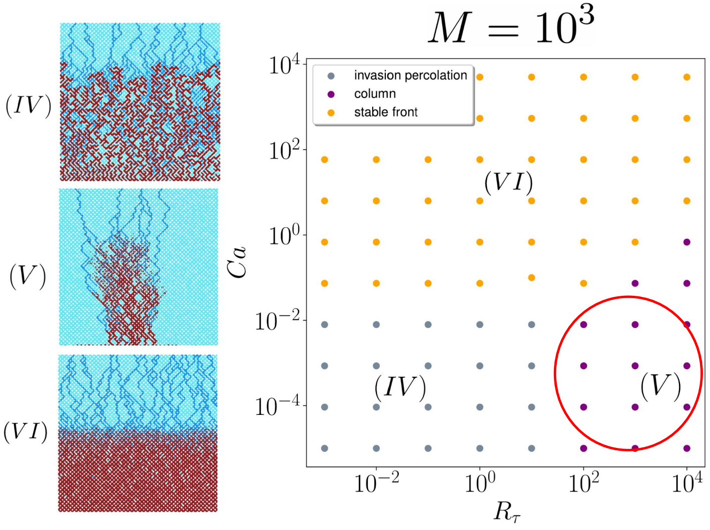


The Column regime

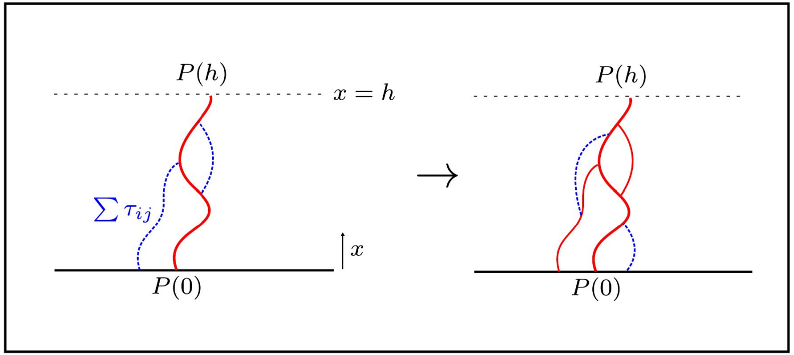
M = Newtonian viscosity / YSF viscosity

$$R_\tau = \frac{L\tau_c}{\sigma}$$





Pressure drop induced by the invasion causes new flowing paths :



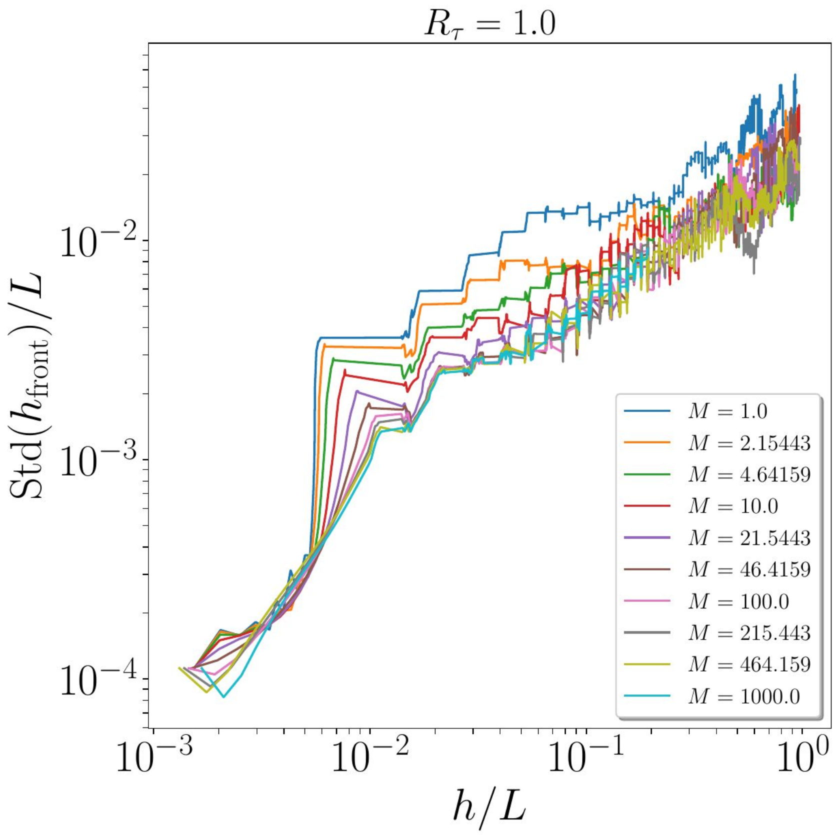
Branching conditions:

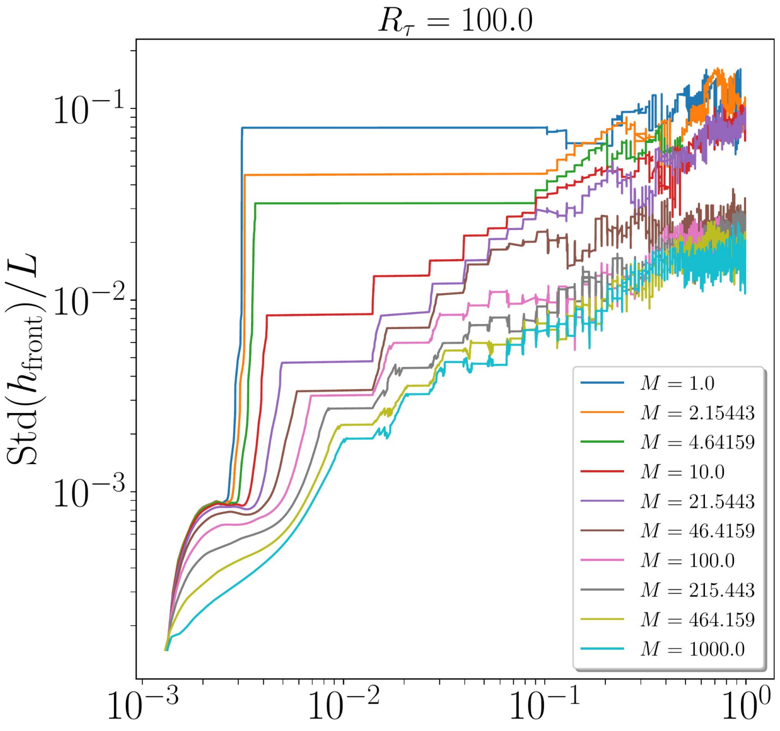
$$\delta P(h) > \langle \tau \rangle h$$

$$\Longrightarrow \ h > L - \frac{M}{\langle \tau \rangle}(\Delta P - \Delta P_0)$$

Appendix

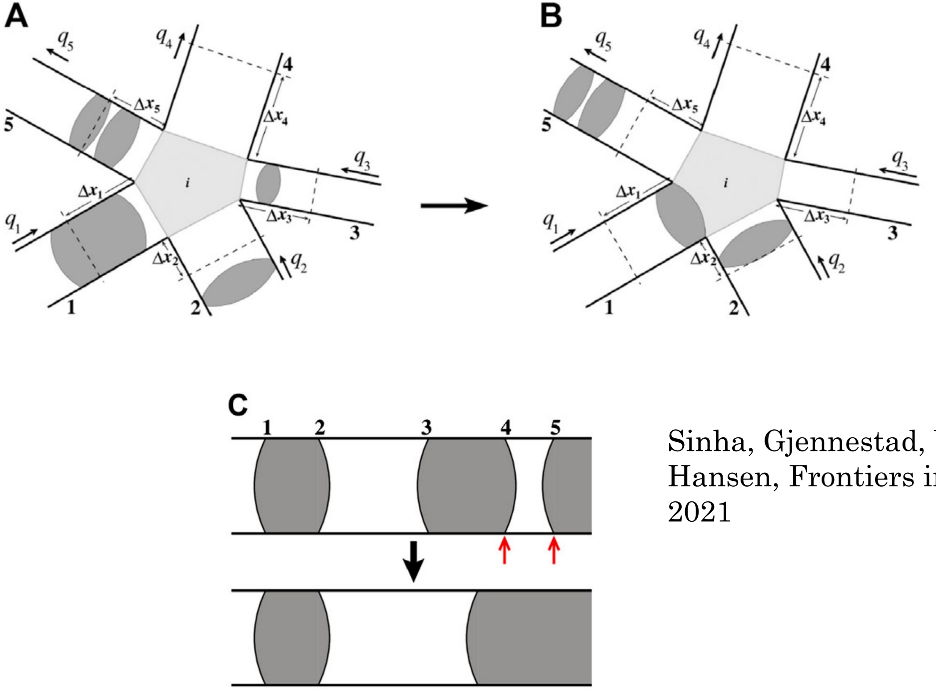
Higher viscosity ratios stabilize the invasion front





The Fluid meniscus algorithms

- Flow rate for each link is determined by solving Kirchhoff equation
- Newtonian fluid enters the pore space and is redistributed proportionally to the flow rate of each link
- The blobs of Newtonian fluid are merged when their number exceeds a threshold value



Sinha, Gjennestad, Vassvik & Hansen, Frontiers in physics, 2021

The Fluid meniscus algorithms

Local flow rate law

$$q_{ij} = \begin{cases} \kappa_{ij}\left(\delta P_{ij} - \tau_{ij} - P^{\sigma}_{ij}\right), & \text{if } \delta P_{ij} - P^{\sigma}_{ij} > \tau_{ij}, \\ 0, & \text{if } |\delta P_{ij} - P^{\sigma}_{ij}| \leq \tau_{ij}, \\ \kappa_{ij}\left(\delta P_{ij} + \tau_{ij} - P^{\sigma}_{ij}\right), & \text{if } \delta P_{ij} - P^{\sigma}_{ij} < -\tau_{ij}, \end{cases}$$

with the capillary term

$$P^{\sigma}_{ij} = \sum_{k=1}^{n} \pm \frac{2\sigma}{r_{ij}} \left[1 - a\cos\left(\frac{2\pi x_k}{l}\right)\right]$$

and the (yield stress) pressure threshold

$$\tau_{ij} = \frac{2\tau_c l_{ij}}{r_{ij}}.$$

Dimensionless time

δt : Time to cross a link of length $\iota = 1$

We define the average velocity $u = 1/\delta t$ and use $u = \dfrac{Q}{\pi r_0^2}$

From Darcy's law :
$$u = \frac{Q}{\pi r_0^2} = \frac{1}{\pi r_0^2}\left(\frac{\pi r_0^4}{8\eta}\frac{\Delta P}{L}\right) = \frac{r^2 \Delta P}{8\eta L}$$

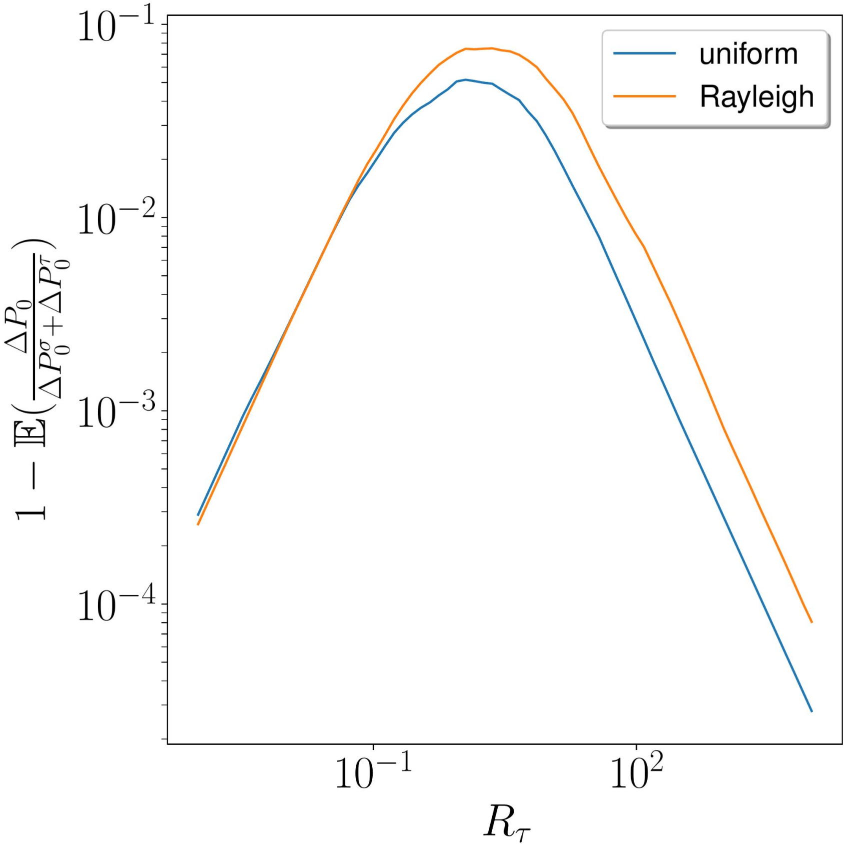
We thus perform the transformation $\delta t \longrightarrow \tilde{\delta t} = \dfrac{8\eta L}{r_0^2 \Delta P}\delta t$

Search for a breakthrough condition : Min-max algorithm

$$\Delta P_0 = \min_{\mathcal{C}} \max_{\langle ij \rangle \in \mathcal{C}} \left[\frac{4a\sigma}{r_{ij}} + \min_{\mathcal{C}' \in \mathcal{S}_{ij}(\mathcal{C})} \sum_{\langle kl \rangle \in \mathcal{C}'} \tau_{kl} \right]$$

$$\Delta P_0(R_\tau = \infty) = \Delta P_0^\sigma \equiv \min_{\mathcal{C}} \sum_{\langle ij \rangle \in \mathcal{C}} \tau_{ij}$$

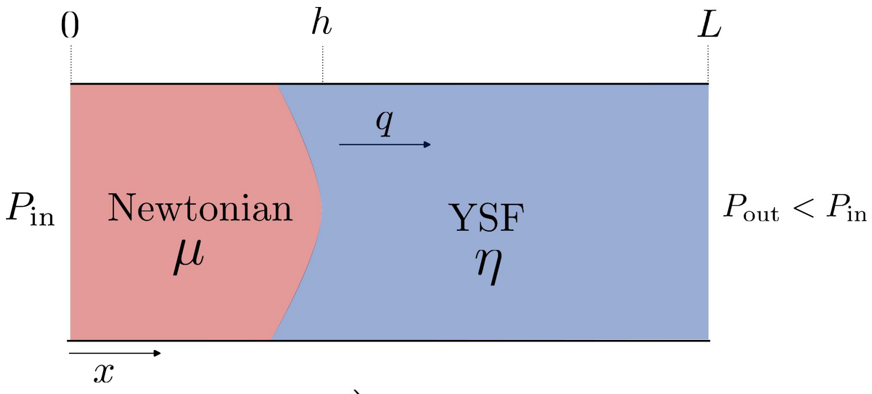
$$\Delta P_0(R_\tau = 0) = \Delta P_0^\sigma \equiv \min_{\mathcal{C}} \max_{\langle ij \rangle \in \mathcal{C}} \frac{4a\sigma}{r_{ij}}$$



1d model enables us to compute the pressure difference as the Newtonian fluid progresses

Pressure drop contributions :

- Inside the Newtonian fluid
- At the interface
- Inside the YSF



Get the flow rate : $q = k\Big(h\mu + (L-h)\eta\Big)^{-1}\Big(|\Delta P| - (L-h)\langle\tau\rangle - 2\sigma\langle 1/r\rangle\Big)$

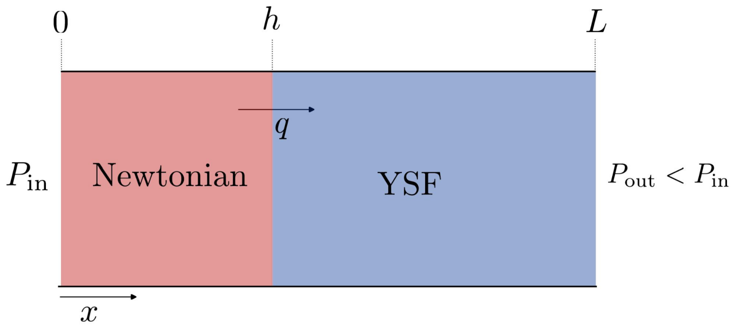
Define $\delta P(h) = P(0) - P(h)$ and use Darcy' law :

$$\Longrightarrow \quad \delta P(h) = -\int_0^h dx\left(-\frac{\mu}{k}q\right) = \mu h\Big(h\mu + (L-h)\eta\Big)^{-1}\Big(|\Delta P| - (L-h)\langle\tau\rangle - 2\sigma\langle 1/r\rangle\Big)$$

$$= \left(1 + \frac{L-h}{h}\frac{1}{M}\right)^{-1}\Big(|\Delta P| - (L-h)\langle\tau\rangle - 2\sigma\langle 1/r\rangle\Big)$$

$$M = \mu/\eta$$

Insights from 1D model with variable radii : Stable front speed



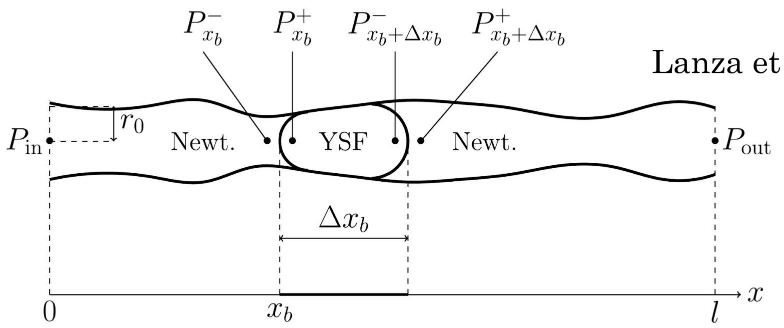
Total pressure drop :

$$\Delta P = \left(\int_0^h dx\, \frac{1}{r(x)^4} \right) \frac{8\mu}{\pi} q + \left(\int_h^L dx\, \frac{1}{r(x)^4} \right) \frac{8\eta}{\pi} q + 2\tau_c \int_h^L dx\, \frac{1}{r(x)} + \frac{2\sigma}{r(h)}$$

$$\frac{dh}{dt} = \frac{1}{8r(h)^2\eta} \underbrace{\left(ML\left\langle \frac{1}{r^4} \right\rangle + (1-M) \int_h^L dx\, \frac{1}{r(x)^4} \right)^{-1}}_{\text{Effective viscosity term}} \left(\Delta P - \underbrace{\int_h^L dx\, \frac{2\tau_c}{r(x)}}_{\text{Yield stress}} - \underbrace{\frac{2\sigma}{r(h)}}_{\text{Capillary term}} \right) \quad M = \mu/\eta$$

Large M limit with uniform radii : $\displaystyle \frac{dh}{dt} = \frac{r_0^2}{8\eta} \frac{1}{Mh} \left(\Delta P - (L-h) \frac{2\tau_c}{r_0} - \frac{2\sigma}{r_0} \right)$

Insights from 1D model : Stable front speed



Lanza et al. (2022)

Limit M = 1 :

$$\frac{dh}{dt} = \frac{1}{8r(h)^2\eta}\underbrace{\left(ML\left\langle\frac{1}{r^4}\right\rangle + (1-M)\int_h^L dx\,\frac{1}{r(x)^4}\right)^{-1}}_{\text{Effective viscosity term}}\left(\Delta P - \underbrace{\int_h^L dx\,\frac{2\tau_c}{r(x)}}_{\text{Yield stress}} - \underbrace{\frac{2\sigma}{r(h)}}_{\text{Capillary term}}\right)$$

$$\frac{dh}{dt} = \frac{1}{8\eta}\frac{1}{L\langle 1/r^4\rangle}\left(\Delta P - \int_h^L dx\,\frac{2\tau_c}{r(x)} - \frac{2\sigma}{r(h)}\right)$$

$$\frac{d^2h}{dt^2} = \frac{1}{4\eta r(h)^2}\frac{1}{L\langle 1/r^4\rangle}\left(\frac{\sigma r'(h)}{r(h)^2} - \frac{\tau_c}{r(h)}\right)\frac{dh}{dt}$$

$$\alpha = 1/(4\eta L\langle 1/r^4\rangle r_0) \approx \frac{r_0^3}{4\eta L}$$

$$r(x) = \frac{r_0}{1 + a\cos\omega h} \implies r(h)^2\frac{dh}{dt} + \frac{2a\alpha}{r_0}\left(\frac{\tau_c}{\omega}\sin(\omega h) + \sigma\big(\cos(\omega h) - 1\big)\right) - \left(u_0 - \frac{2\alpha\tau_c}{r_0}h\right) = 0$$

The two extreme cases give an upper bound of the critical pressure threshold

$$\Delta P_0 = \min_{\mathcal{C}} \max_{\langle ij \rangle \in \mathcal{C}} \left[\frac{4a\sigma}{r_{ij}} + \min_{\mathcal{C}' \in \mathcal{S}_{ij}(\mathcal{C})} \sum_{\langle kl \rangle \in \mathcal{C}'} \tau_{kl} \right]$$

Let :

$$\Delta P_0^{\tau} = \min_{\mathcal{C}} \sum_{\langle ij \rangle \in \mathcal{C}} \tau_{ij}$$ the yield-stress only pressure threshold

$$\Delta P_0^{\sigma} = \min_{\mathcal{C}} \max_{\langle ij \rangle \in \mathcal{C}} \frac{4a\sigma}{r_{ij}}$$ the capillary pressure threshold

One can check that $\Delta P_0 \leq \Delta P_0^{\sigma} + \Delta P_0^{\tau}$