Chaotic Advection is Inherent to Heterogeneous Darcy Flow

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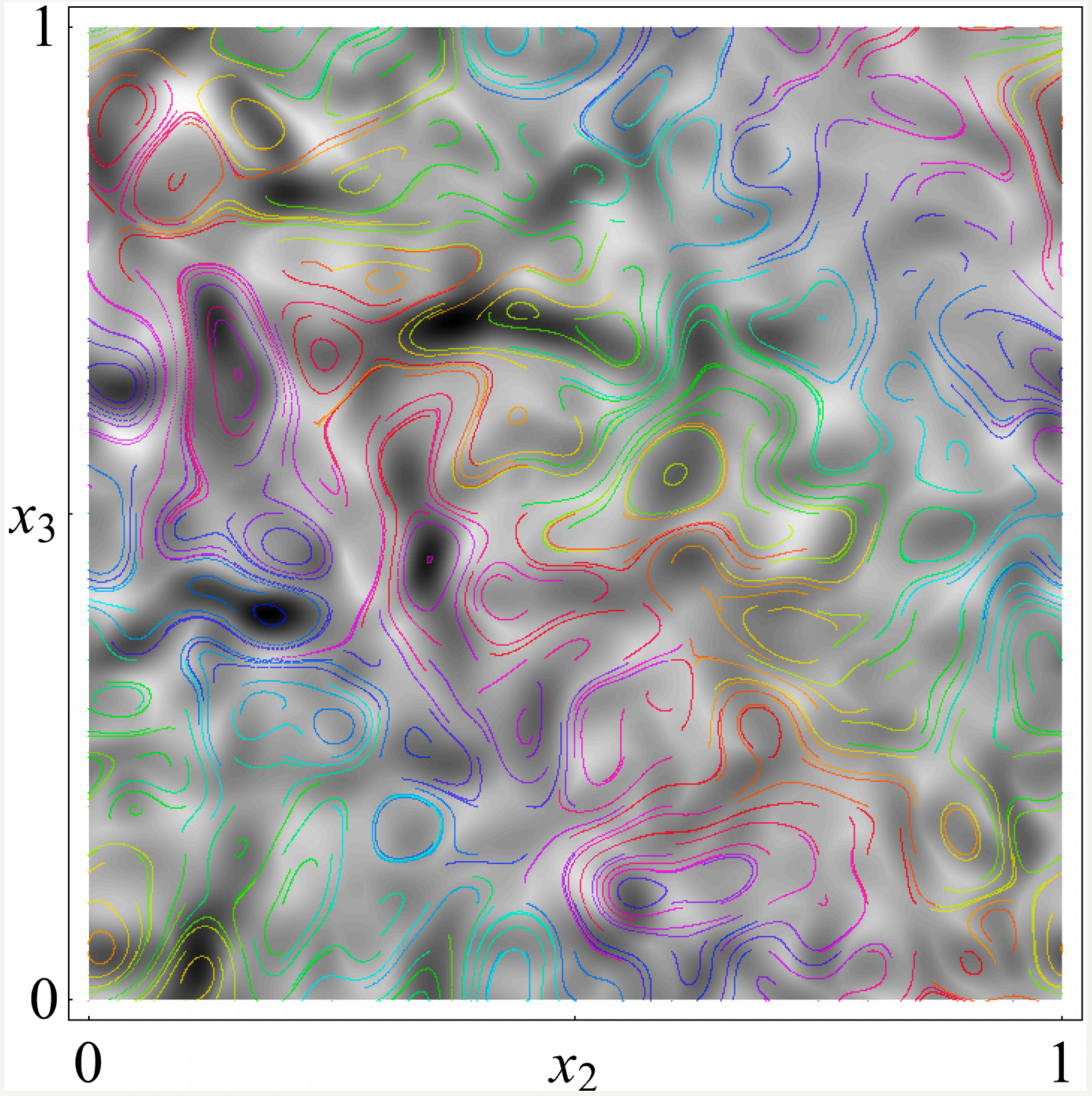
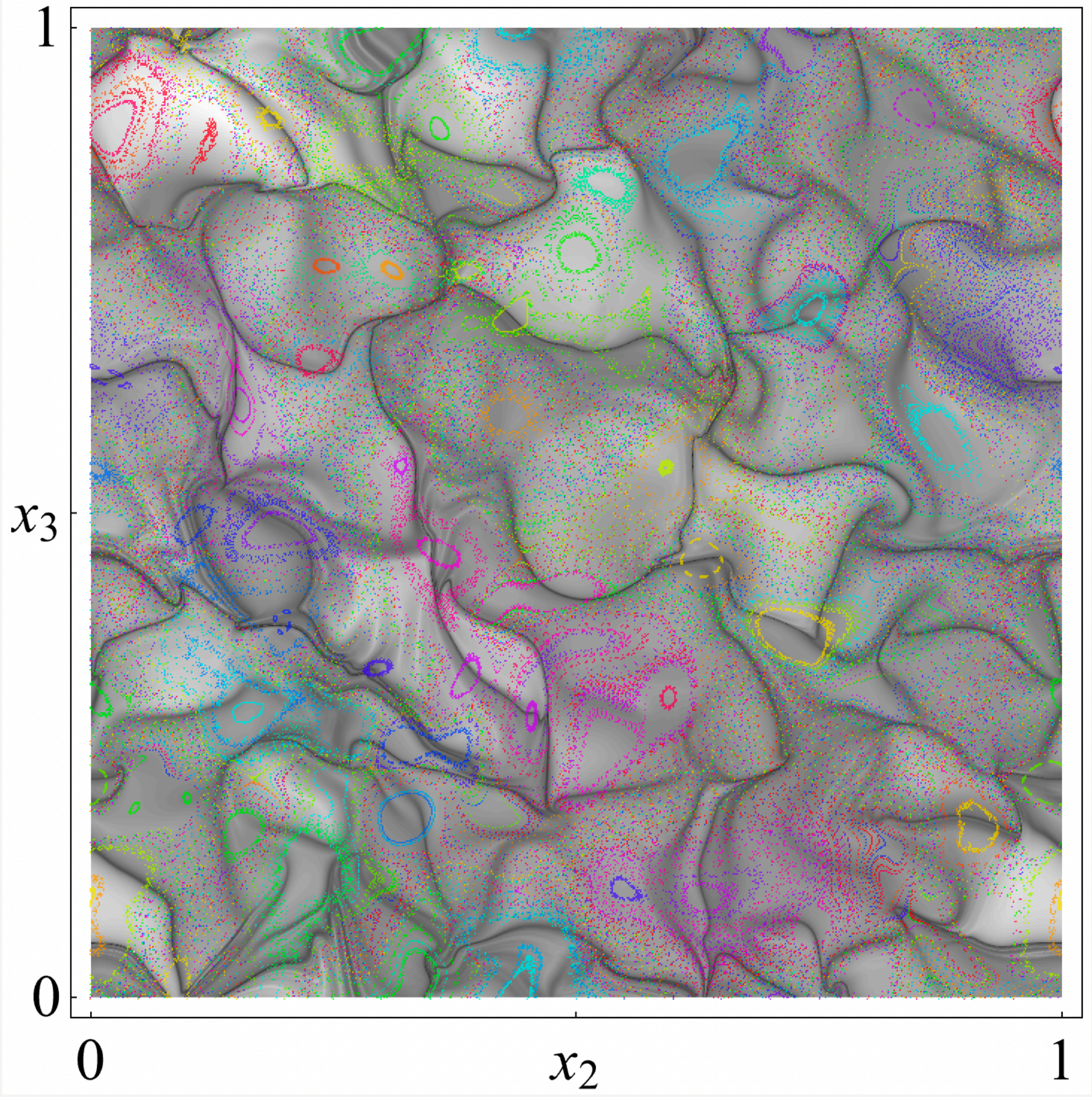
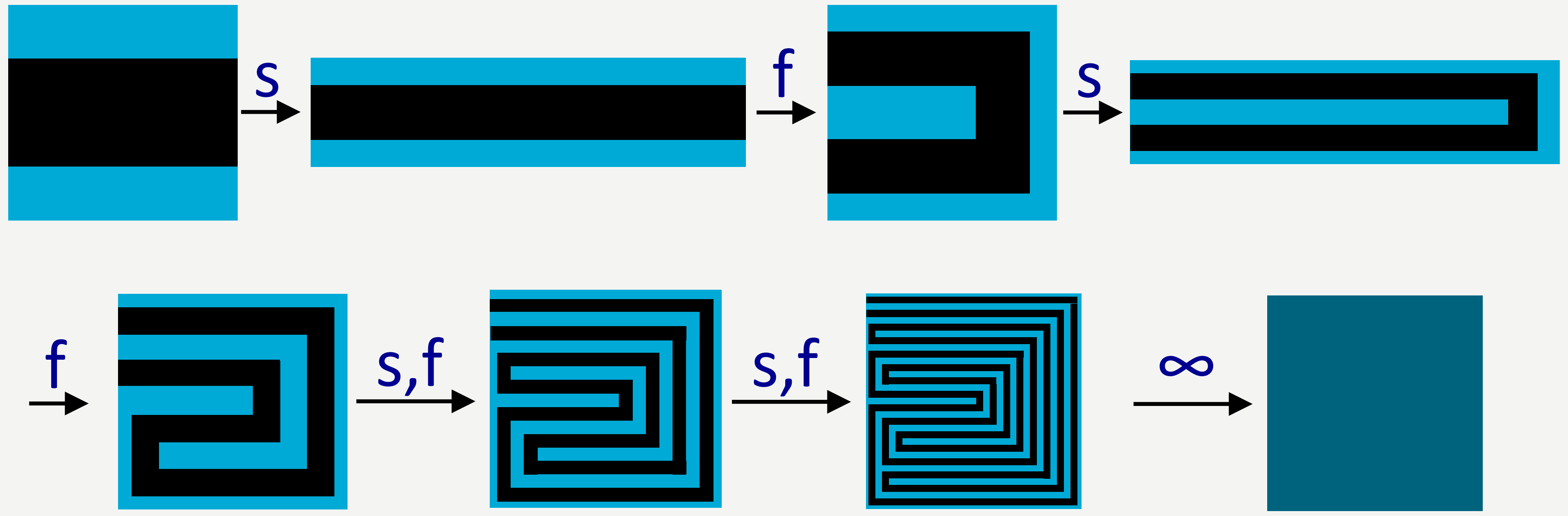


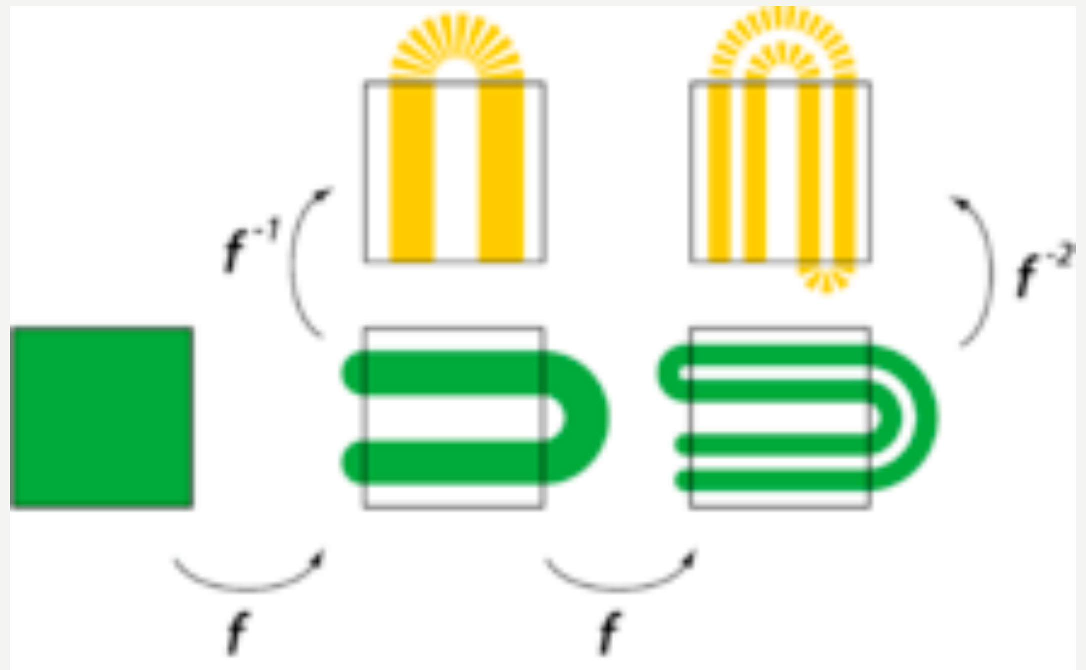


Interpore 2026, Nantes, France May 19-22

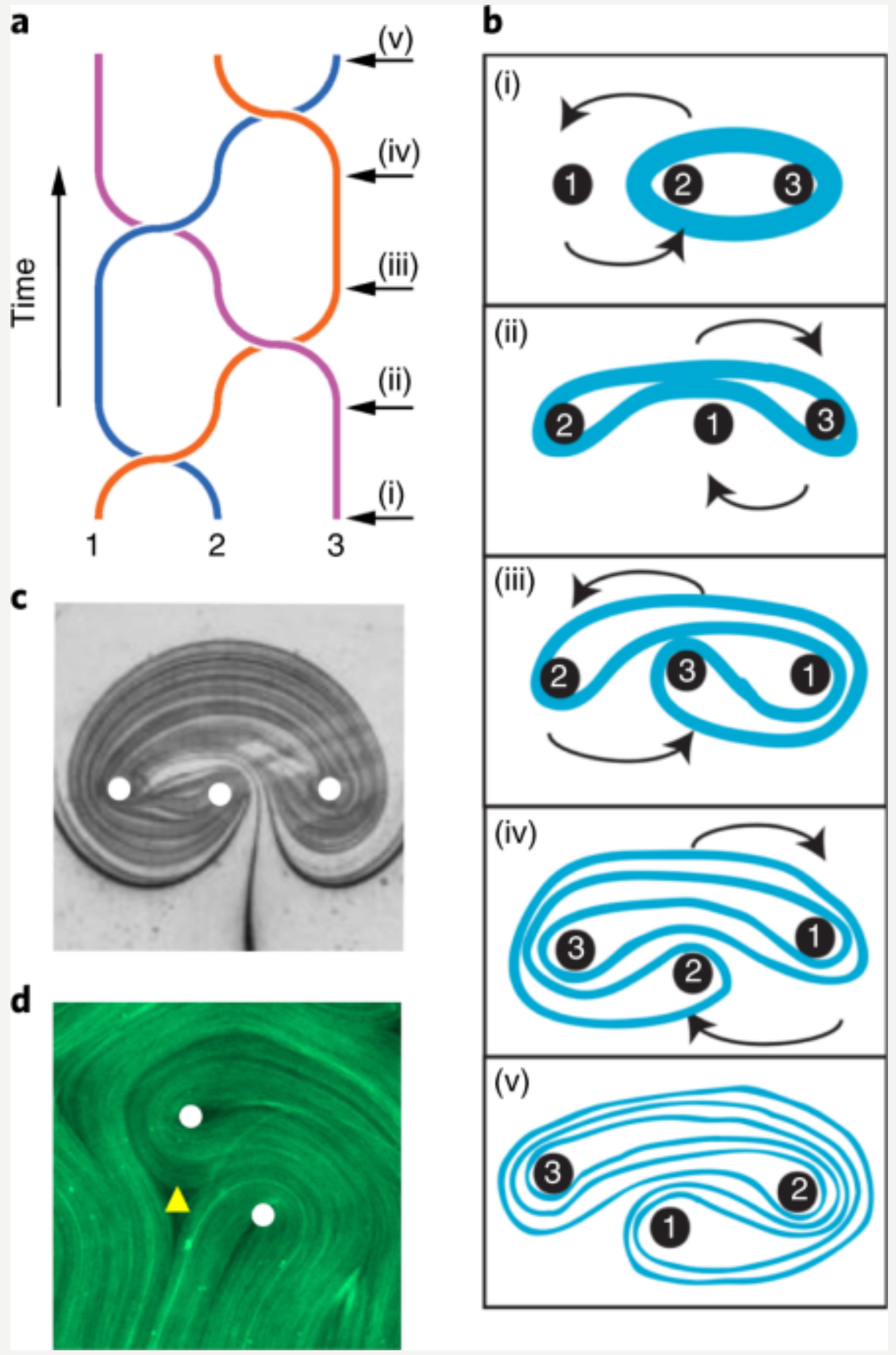
What is chaotic advection?

- Advection equation is a dynamical system rich enough to admit chaos: $\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t)$
- Chaotic mixing arises if d.o.f ≥ 3 (2D unsteady, 3D steady), particle trajectories form chaotic orbits:
- Stretching and folding motions leads to exponential stretching: $l(t) = l(0)e^{\lambda_\infty t}$





Smale horseshoe map

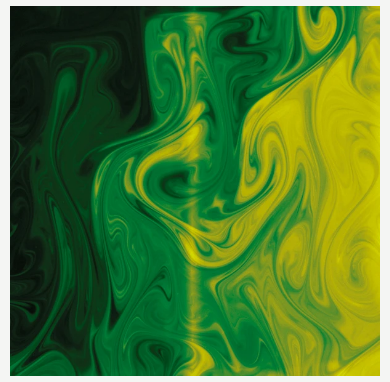


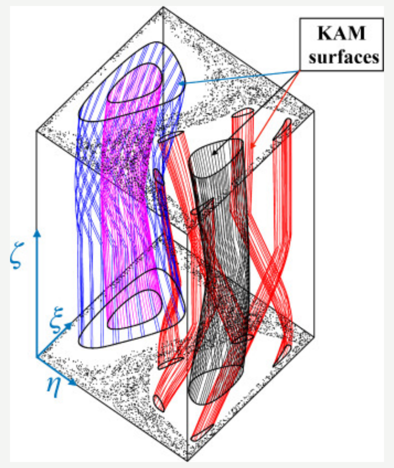
Braiding and exponential stretching in nematic liquid crystals
Tan et al, Nature Phys. (2019)

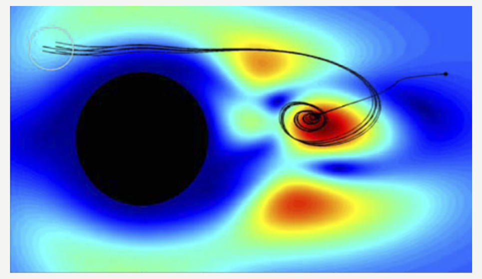
- Strength of chaos characterised by (infinite time) Lyapunov exponent λ_∞

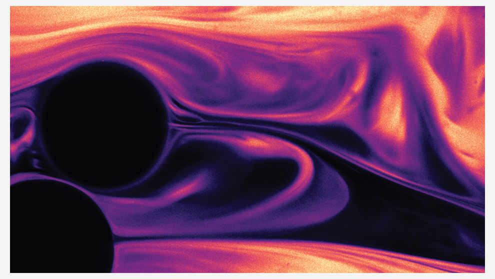
Why does it matter?

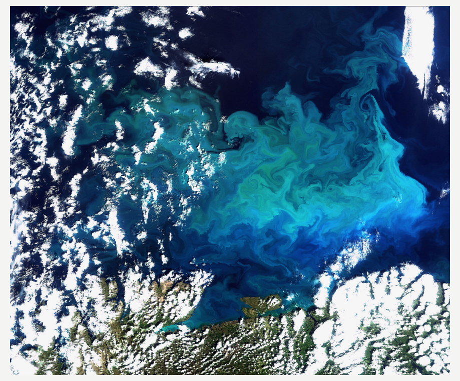
- Chaotic mixing **fundamentally alters** many fluid-borne processes in porous media:
  - Solute mixing - mixing rates are exponential and singular:
  - Solute dispersion - transverse (longitudinal) dispersion enhanced (retarded):
  - Colloid transport - hydrodynamic trapping, deposition “hot spots”:
  - Chemical reactions - accelerated kinetics, augmented stability:
  - Biological activity - competitive coexistence, singular kinetics:











Aref et al, Rev. Mod. Phys. (2017)

Rothstein et al, Nature (1999)

Jones, Phys. D. (1999)

Haller and Sapsis, Phys. D. (2008)

Tel et al, Rev. Mod. Phys. (2005)

Karolyi et al, Phys. Rev. Lett. (2000)

- Need to properly resolve the advective dynamics to understand, quantify and predict these phenomena

Isotropic 3D Darcy flow

- Isotropic Darcy flow has zero helicity density h (a measure of topological complexity):

$$\mathbf{v}(\mathbf{x}) = -k(\mathbf{x})\nabla\phi \qquad h \equiv \mathbf{v}\cdot(\nabla\times\mathbf{v}) = k\nabla\phi\cdot(\nabla\phi\times\nabla k) = 0$$

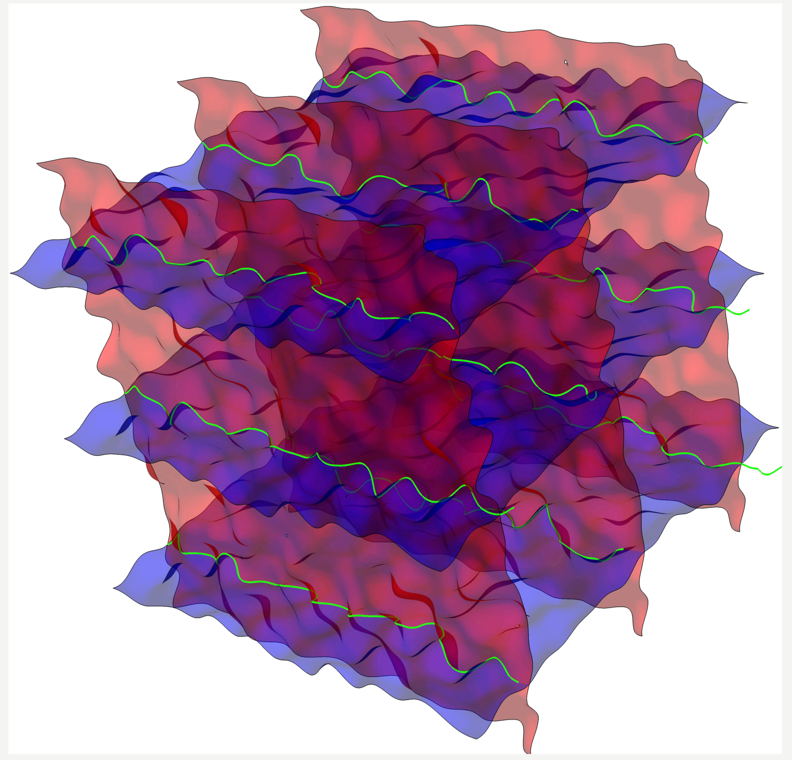
Moffatt, J. Fluid Mech. (1969)
Sposito, Adv. Wat. Res. (2001)

- Thus isotropic Darcy flow admits a pair of streamfunctions: $\mathbf{v} = \nabla\times\mathbf{A} = \nabla\psi_1\times\nabla\psi_2$

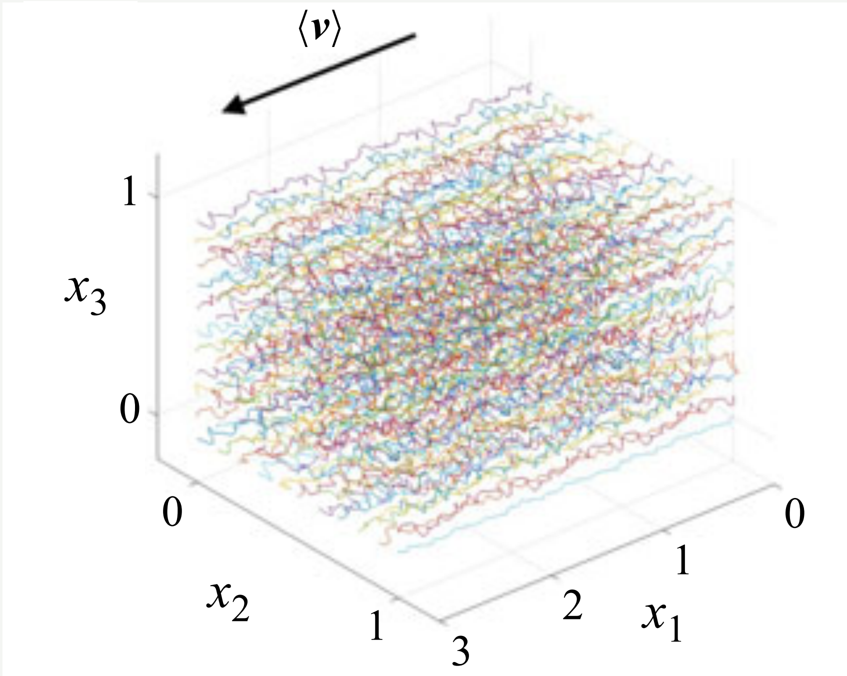
Yoshida, J. Math. Phys. (2009)

- Streamfunctions constrain Lagrangian kinematics: Lester et al, J. Fluid Mech. (2021, 2022), Wat. Resour. Res. (2023)
  - no streamline braiding:
  - non-chaotic mixing: $\lambda_\infty = 0 \qquad l(t) \sim t^r$
  - zero advective transverse macrodispersion: $D_T^m = 0$

- Even if strongly heterogeneous*, isotropic Darcy flows are non-chaotic (*if k is smooth)



Zero helicity isotropic Darcy flow
Lester et al, J. Fluid Mech. (2021)



Zero transverse macrodispersion
Lester et al, Wat. Resour. Res. (2023)

Anisotropic 3D Darcy flow

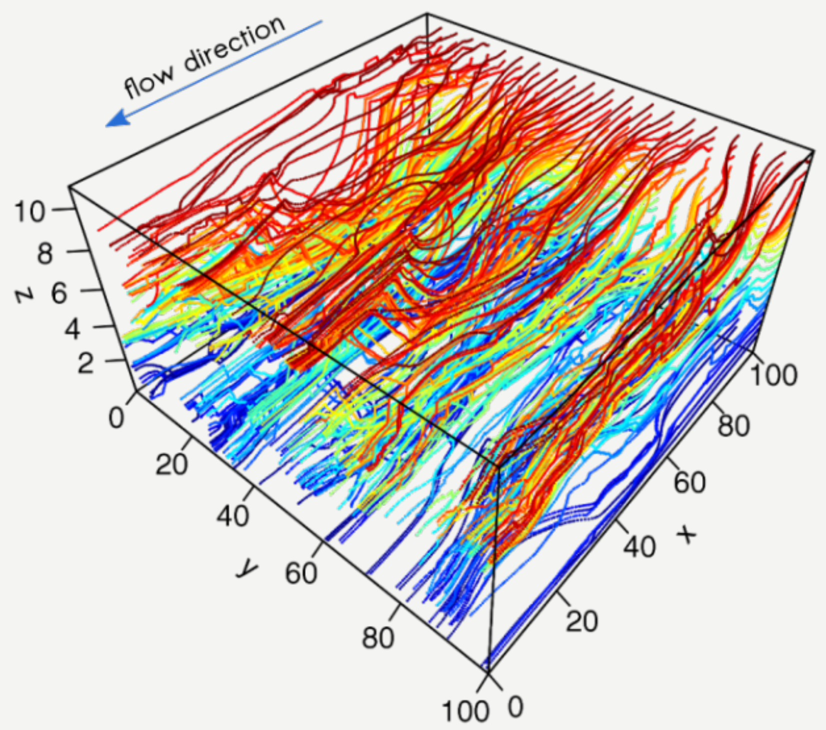
- Anisotropic Darcy flow has non-zero helicity density h, streamsurfaces no longer exist:

$$\mathbf{v}(\mathbf{x}) = -\mathbf{K}(\mathbf{x}) \cdot \nabla\phi, \quad \mathbf{K}(\mathbf{x}) \neq k(\mathbf{x})\mathbf{I} \qquad\qquad h = \mathbf{K} \cdot \nabla\phi \cdot (\nabla \times \mathbf{K} \cdot \nabla\phi) \neq 0$$

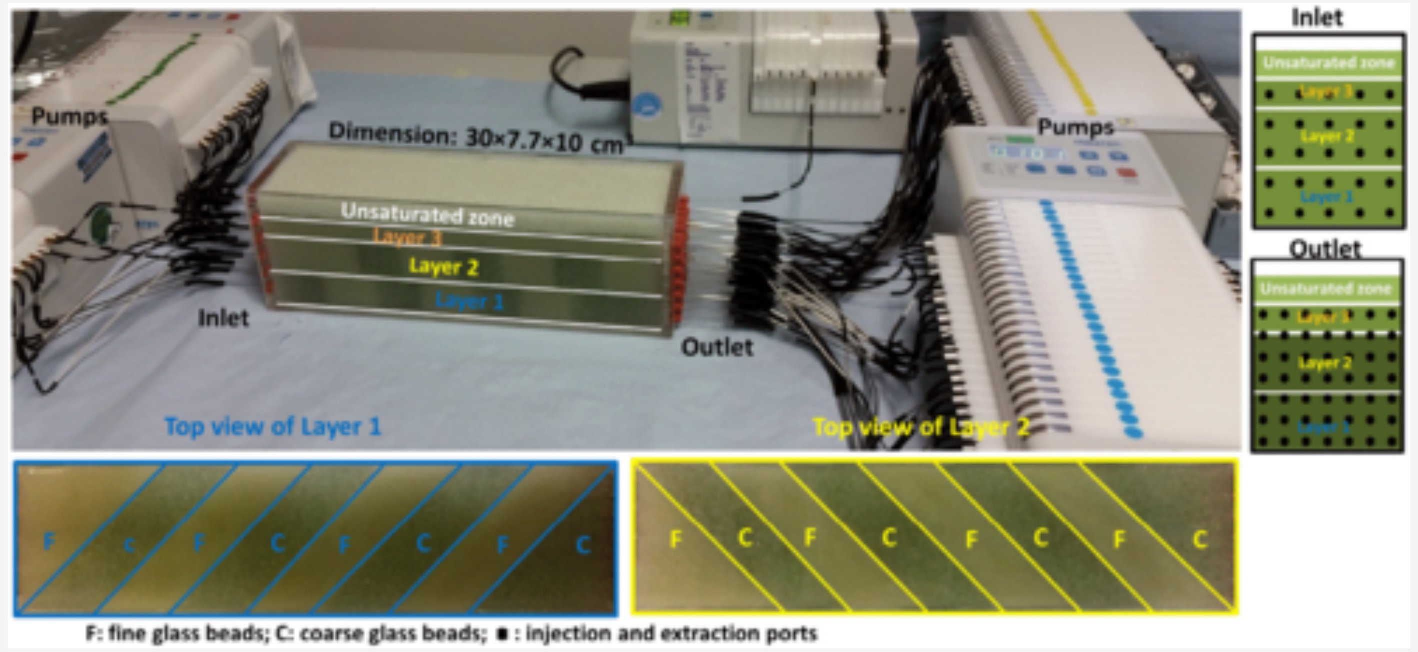
- These flows appear to admit mixing, braiding and transverse dispersion:

- Q: Are these flows chaotic?

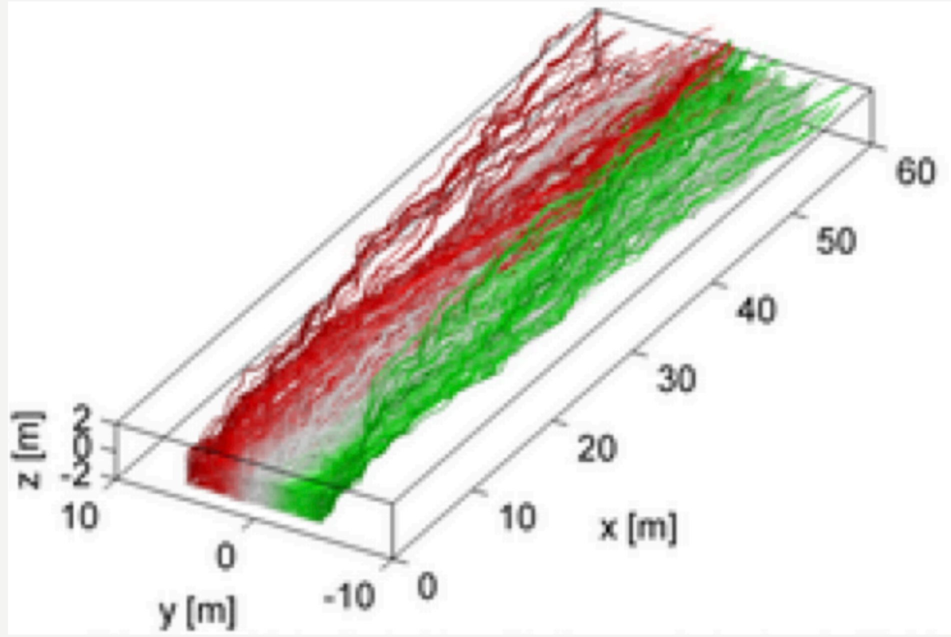
- Q: How prevalent are anisotropic conductivity fields?



Anitostopric Darcy flow
Huber et al, Hydrol. Earth Syst. Sci (2016)

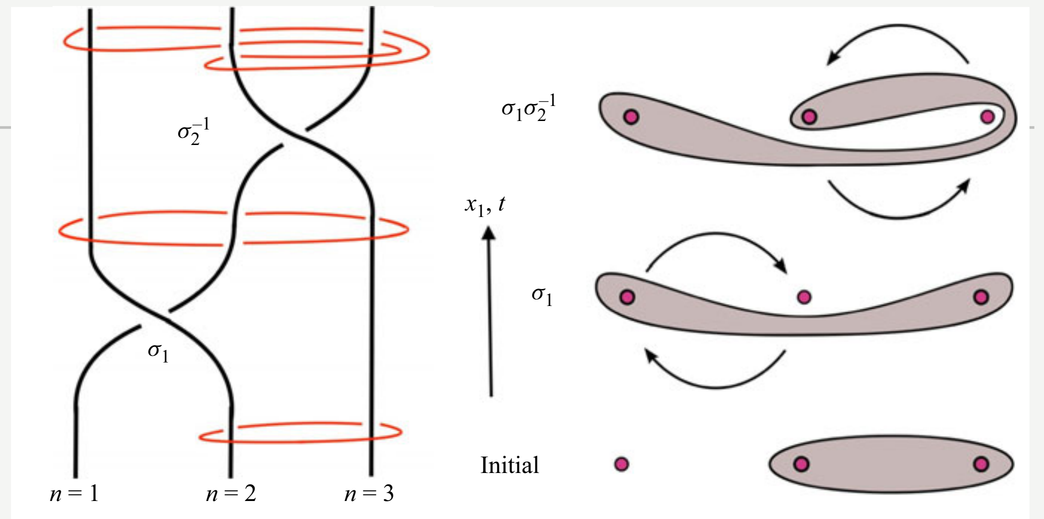


Helical flow in layered porous media: Ye et al, Phys. Rev. Lett. (2015)



Anitostopric Darcy flow
Cirpka et al, Wat. Resour. Res. (2015)

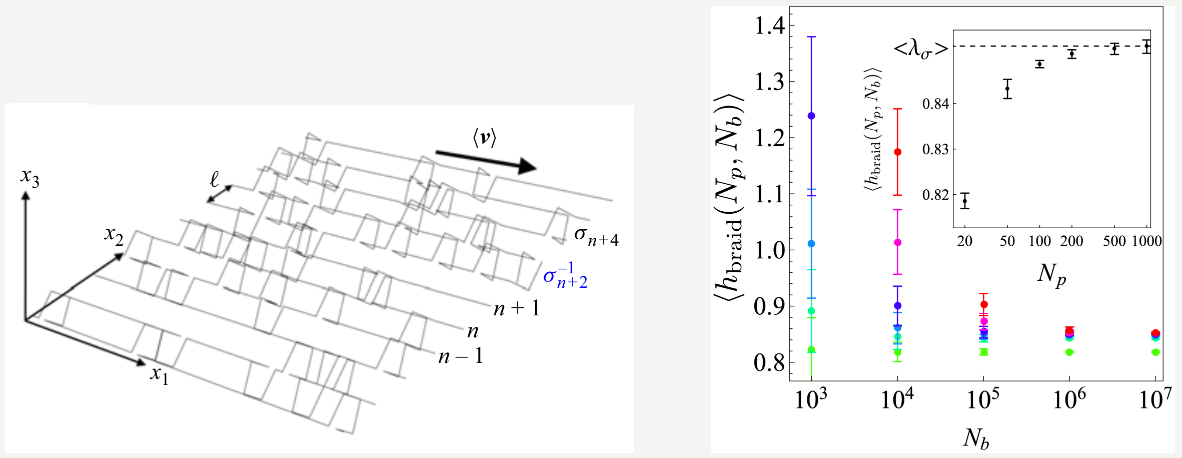
Linking stirring and dispersion



- 3D steady flow, 1D streamlines are **topological obstacles** - if streamlines lines braid, fluid elements must be stretched exponentially:

- Braid word **b** characterises sequences of crossings $\sigma_i^{\pm}$, can estimate Lyapunov exponent:

e.g. $\mathbf{b} = \sigma_1^+ \sigma_2^- \sigma_1^+$ $\qquad$ $\lambda_\infty \approx \lim_{k\to\infty} \frac{1}{k} \ln \frac{|\mathbf{b}^k \ell_E|}{|\ell_E|}$

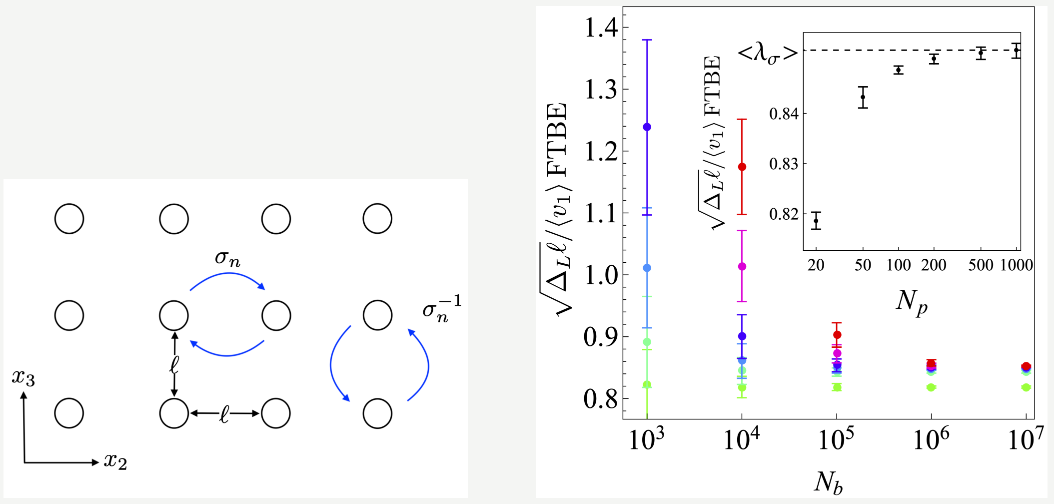


d=1

- We derive a fundamental link between stirring and transverse dispersion in random flows:

Lester et al, ArXiv (2026)

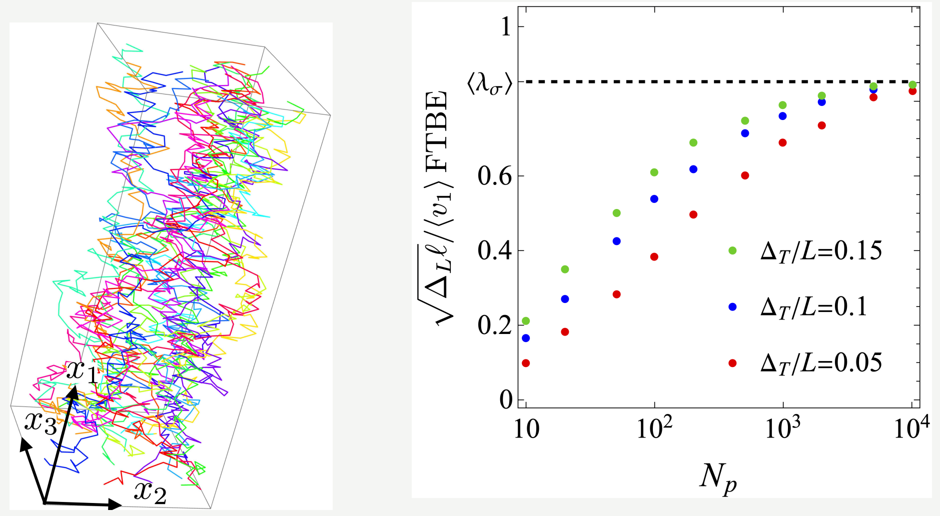
$$\lambda_\infty = \left(\frac{d}{Pe_T}\right)^{1/d} \langle \lambda_\sigma \rangle, \quad Pe_T \equiv \frac{\ell \langle v_1 \rangle}{D_T^m} \qquad \langle \lambda_\sigma \rangle \approx 0.825$$



d=2

- Field data suggests $D_T^m \neq 0$, hence Darcy flow must also be chaotic: $\lambda_\infty > 0$

Zech et al, Groundwater (2019)



d=2

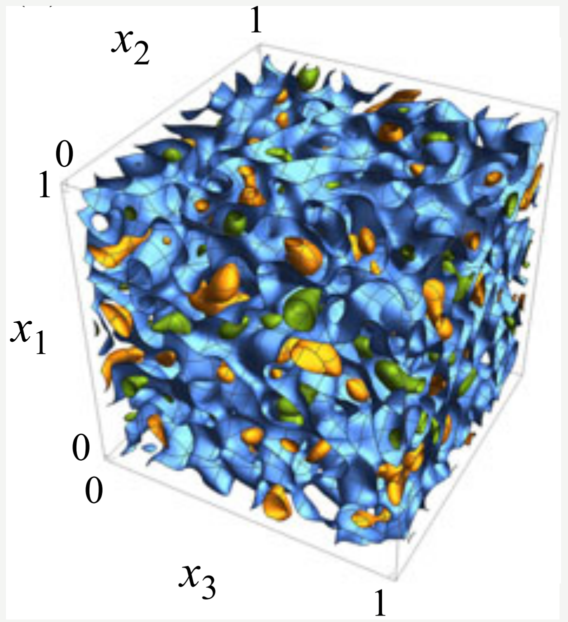
Impact of anisotropy

- We consider anisotropic heterogeneous Darcy flow in triply-periodic cube (3-torus):

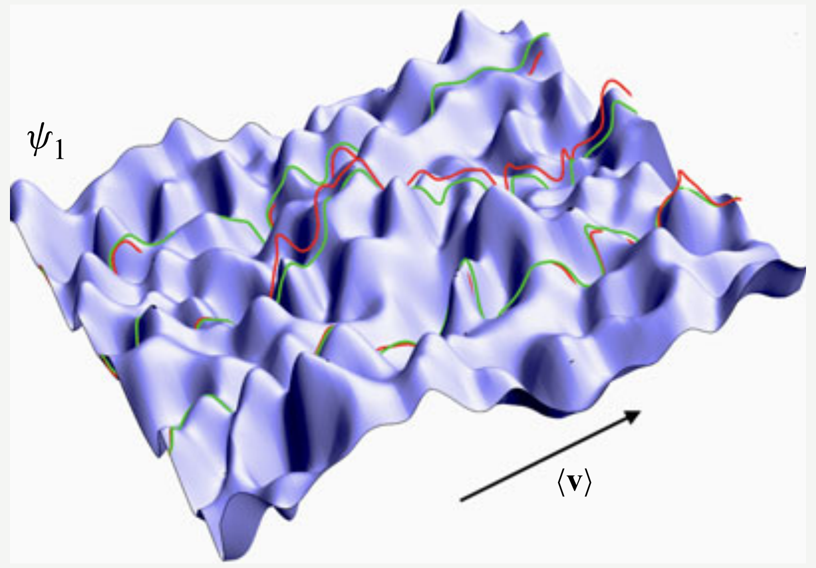
$$\mathbf{v}(\mathbf{x}) = -\mathbf{K}(\mathbf{x}) \cdot \nabla\phi, \quad \mathbf{K}(\mathbf{x}) = k_0(\mathbf{x})\mathbf{I} + \delta[k_\delta(\mathbf{x}) - k_\delta(\mathbf{x})]\, \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1$$

- 3D periodic log-Gaussian random fields $k_0(\mathbf{x}),\ k_\delta(\mathbf{x})$ - two control parameters:

Heterogeneity: $\sigma^2_{\ln K} \in [0,4]$ Anisotropy: $\delta \in [0,1]$

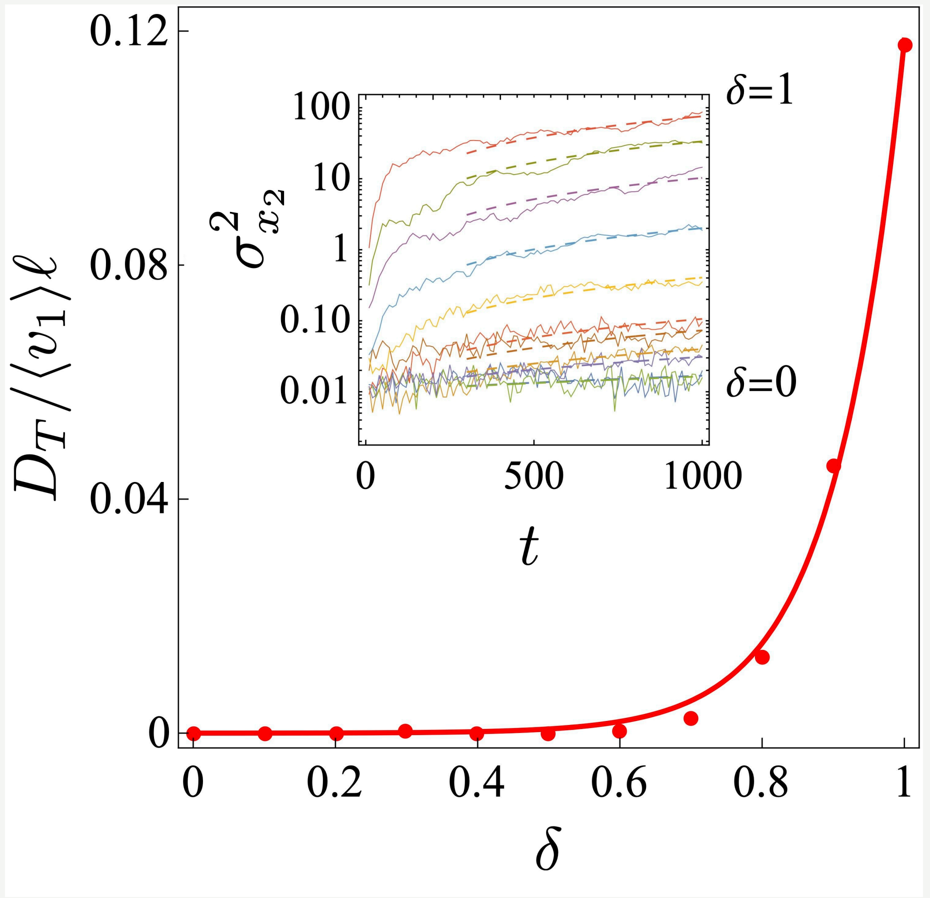


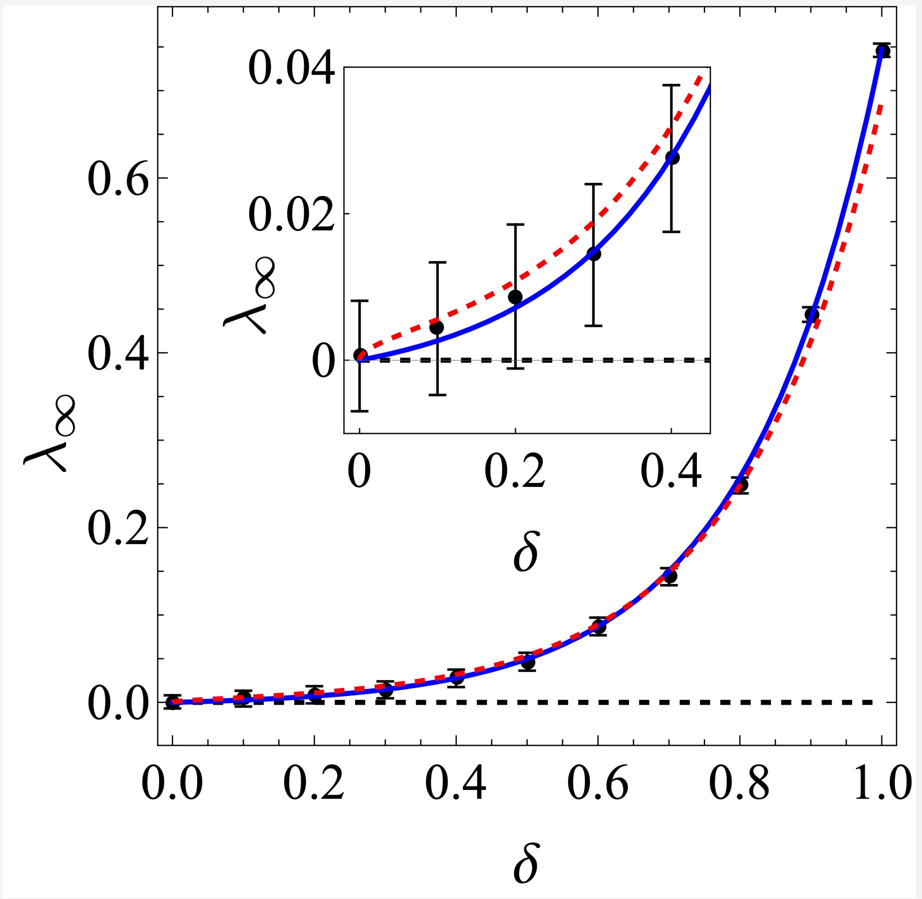
$k_0(\mathbf{x}),\ k_\delta(\mathbf{x})$

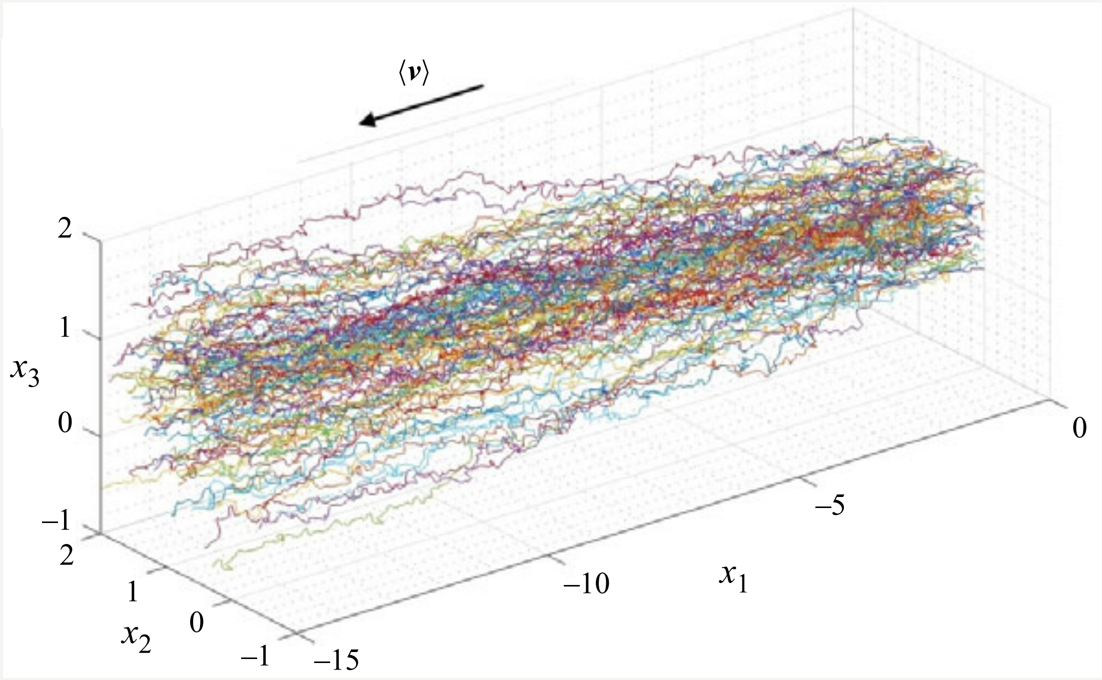


Streamlines and confining streamsurfaces

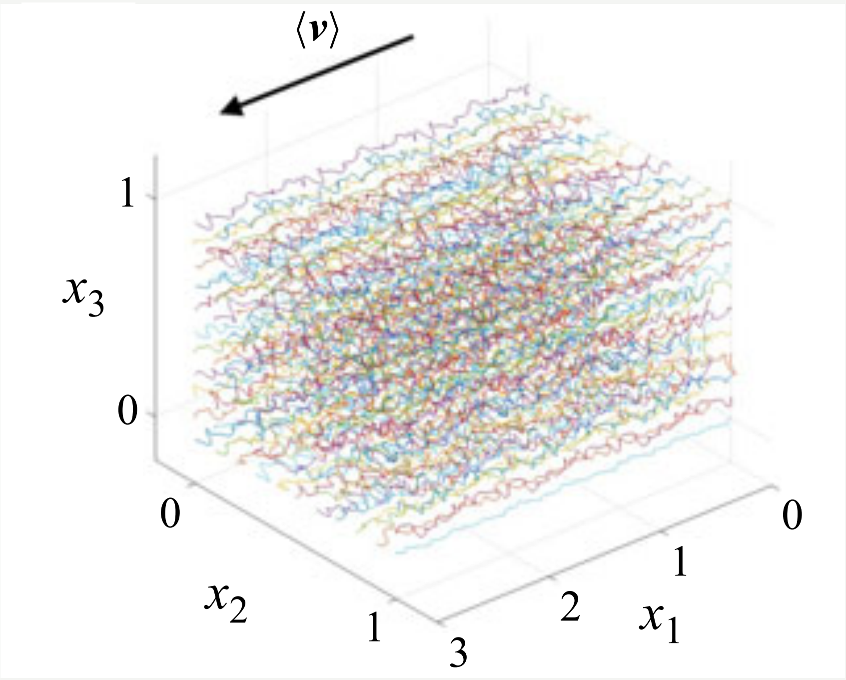
- Increasing anisotropy increases Lyapunov exponent and dispersion:







$\sigma^2_{\ln K} = 4,\ \delta = 1$



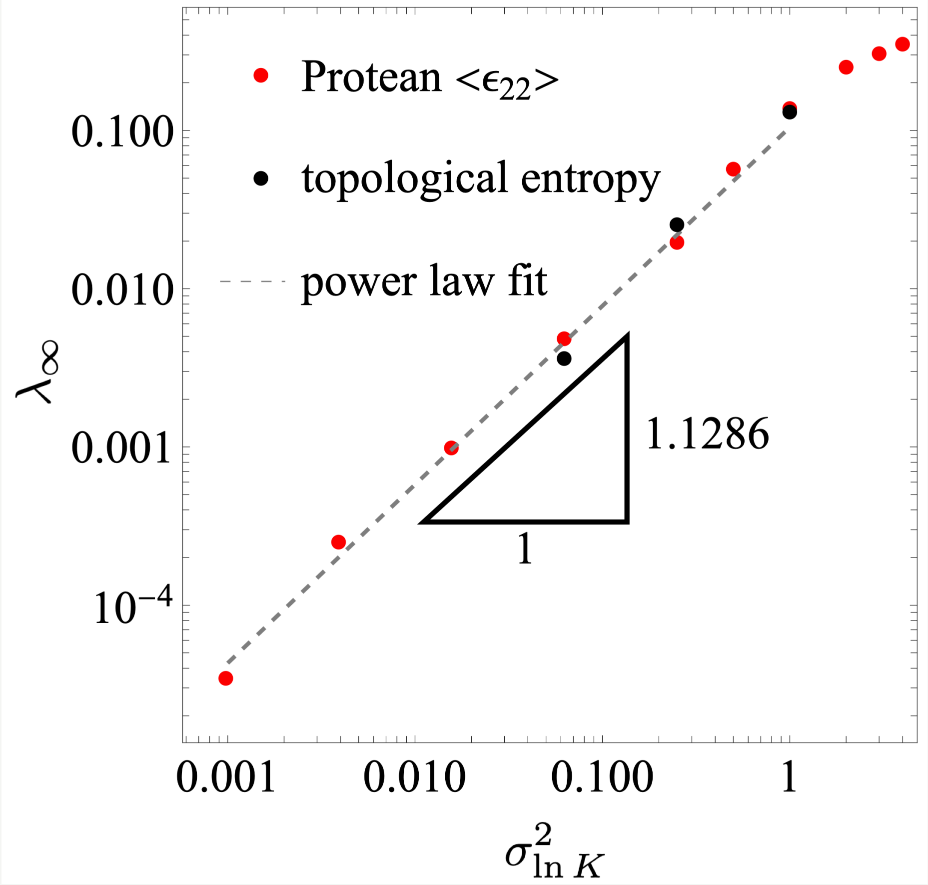
$\sigma^2_{\ln K} = 4,\ \delta = 0$

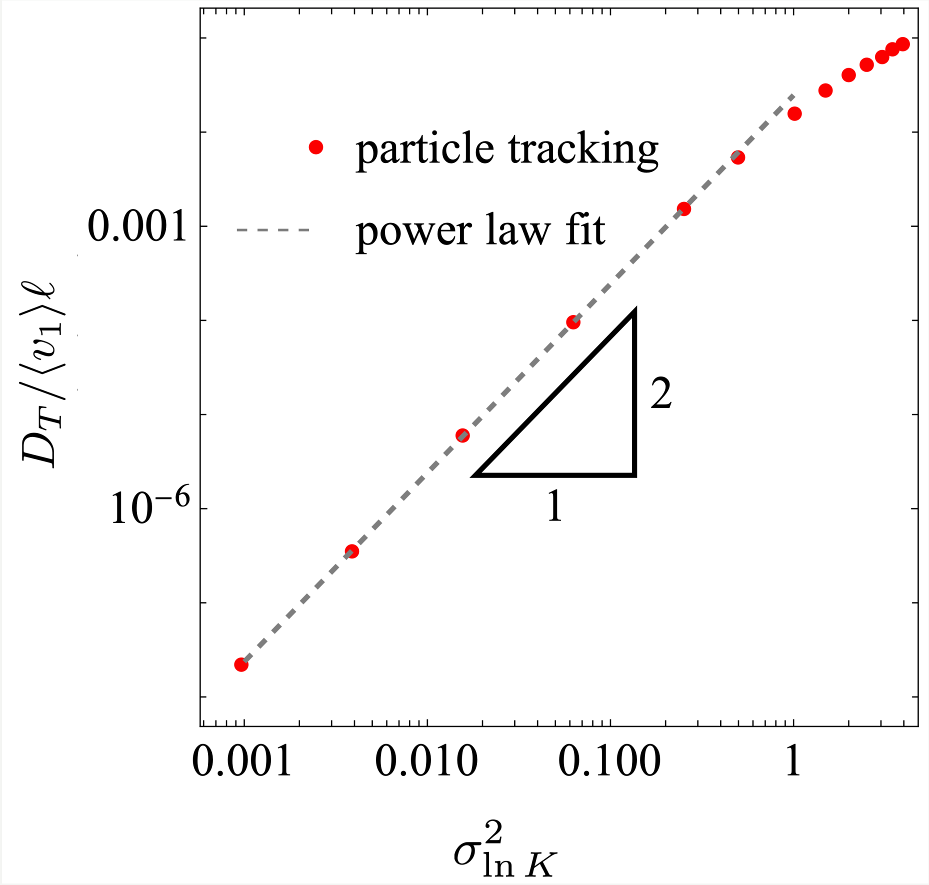
- Braiding theory gives excellent agreement: $\lambda_\infty = \left(\frac{d}{Pe_T}\right)^{1/d} \langle \lambda_\sigma \rangle, \quad Pe_T \equiv \frac{\ell \langle v_1 \rangle}{D_T^m}$

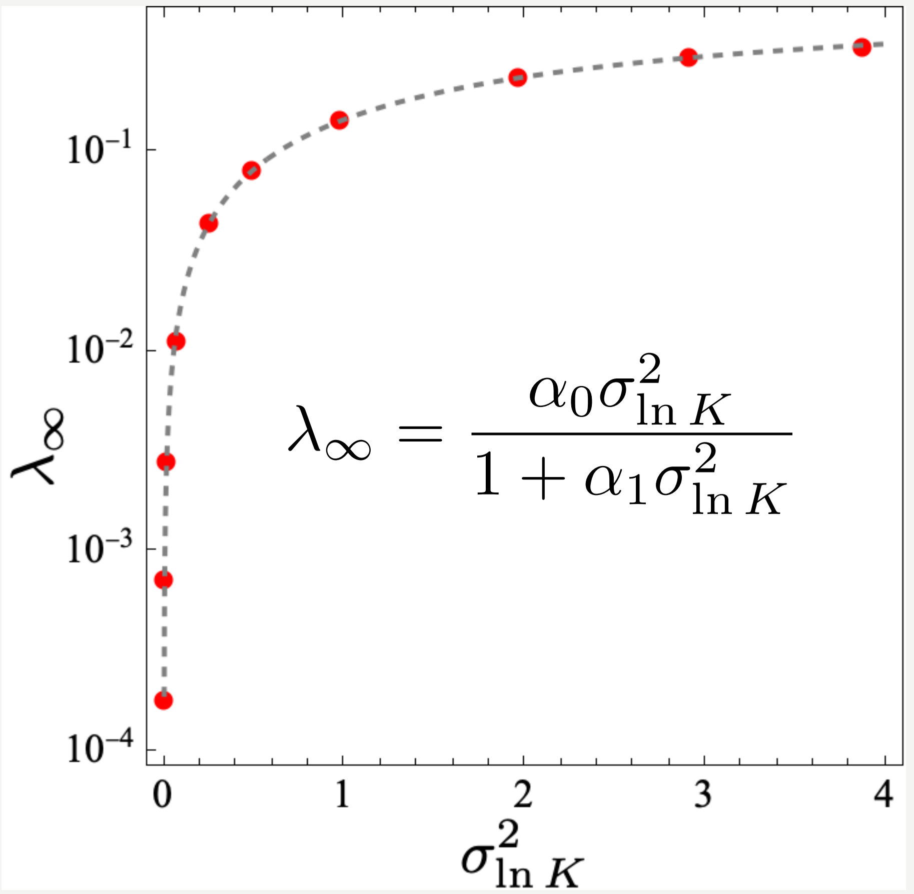
Lester et al, J. Fluid Mech. (2025)

Impact of heterogeneity

- Increasing heterogeneity imparts classical route to Hamiltonian chaos:
- Increasing heterogeneity increases Lyapunov exponent and dispersion:





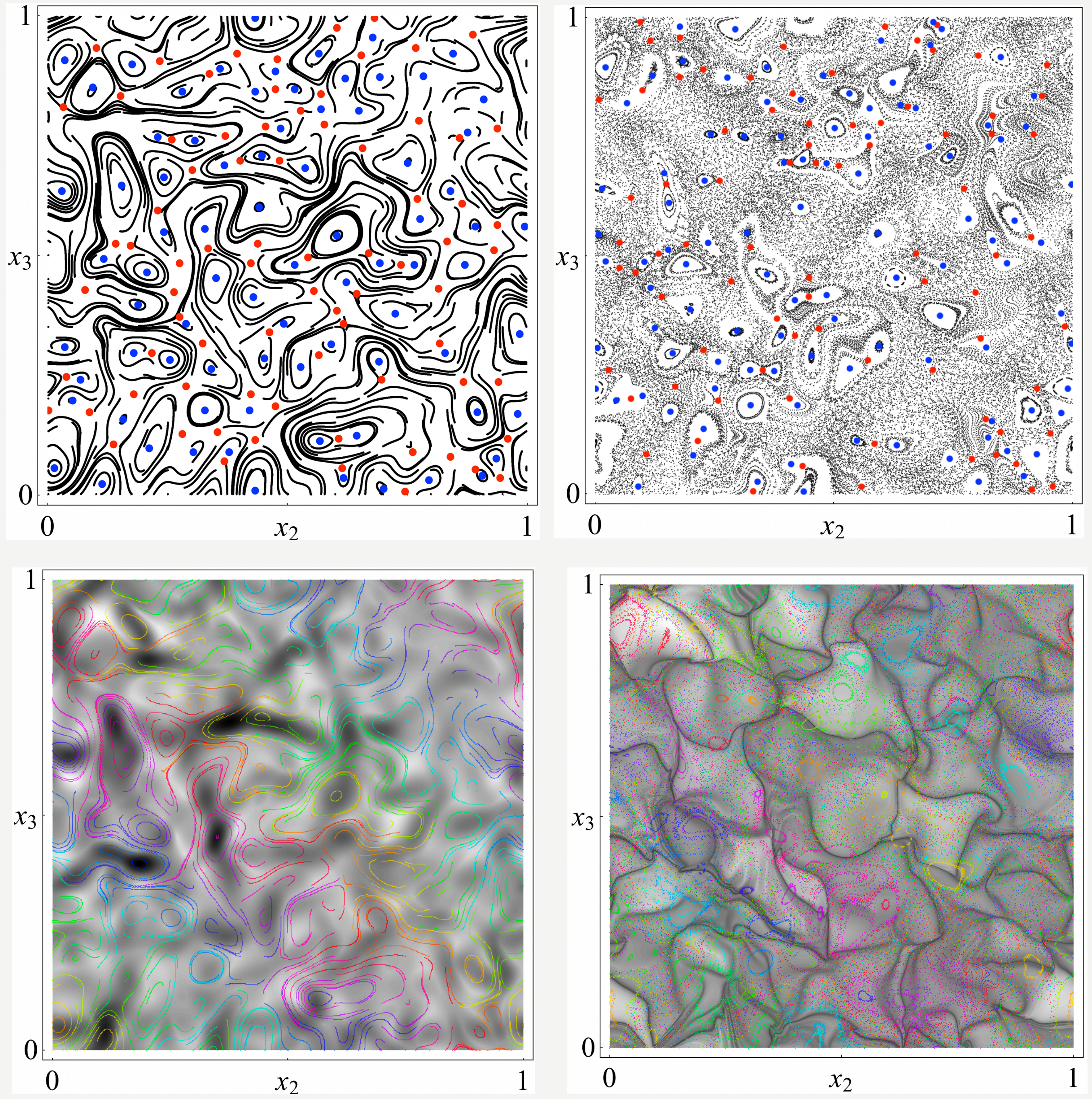


- Efficient mixing in limit of strongly heterogeneous media:

$$\lim_{\sigma^2_{\ln K}\to\infty} \lambda_\infty = \frac{\alpha_0}{\alpha_1} = 0.677 \approx \ln 2$$

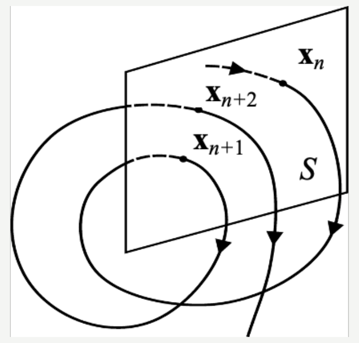
KAM islands & hyperbolic points:

FTLE ridges



$\sigma^2_{\ln K} = 2^{-4}$ $\sigma^2_{\ln K} = 1$

Poincaré sections:

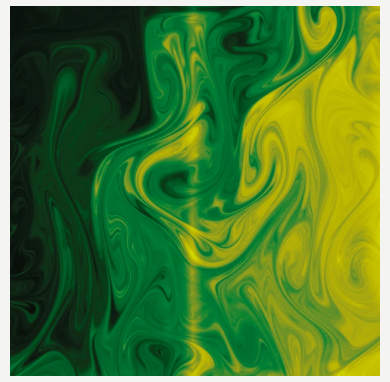


Lester et al, J. Fluid Mech. (2025)

Impacts on fluid-borne processes

- Significant impacts on fluid-borne processes, including:

$$\langle c^2 \rangle \sim \left(\frac{\lambda_\infty}{6t}\right)^{1/4} e^{-\lambda_\infty t/3}$$

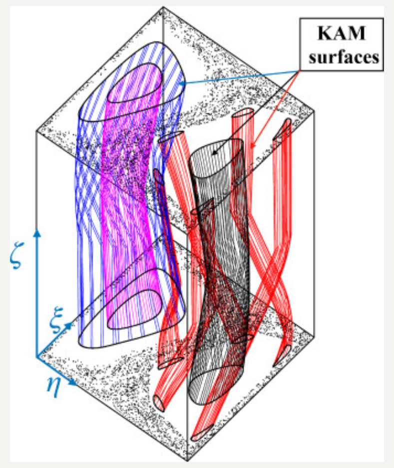


1. Solute mixing - variance decays accelerates from algebraic to exponential:

$$c_m \propto \frac{1}{E(t)} \propto e^{-\lambda_\infty t}$$

2. Transverse solute dispersion - exponentially enhanced:

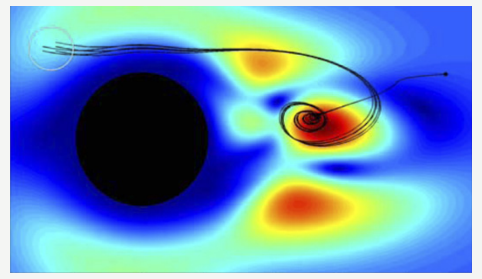
$$D_T = D_{T,\infty} + D_{T,p} \frac{\sinh(2\lambda_\infty t_d)}{2\lambda_\infty t_d}$$



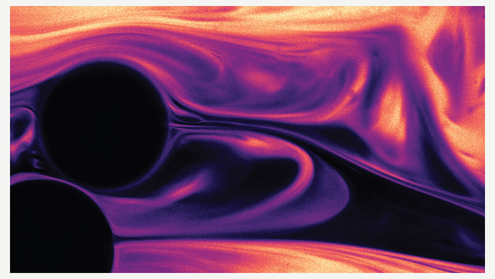
3. Longitudinal dispersion is retarded:

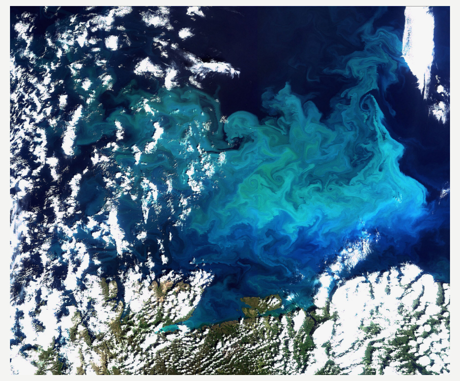
$$\sigma_L^2(t) = \frac{t^2}{(\ln t)^3}$$

4. Trapping of colloidal particles



5. Chemical reactions and biological activity: accelerated reactions and kinetics of more complex reaction systems fundamentally altered

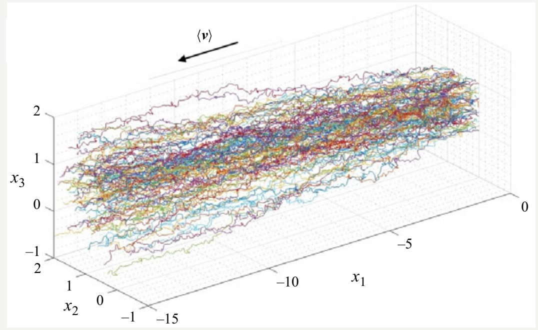




Lester et al, J. Fluid Mech. (2025)

Conclusions

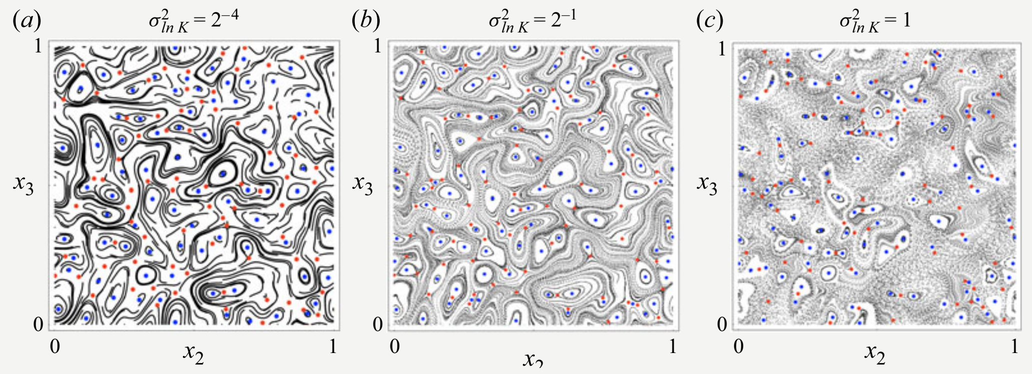
- Chaotic mixing is inherent to heterogeneous Darcy flow due to streamline braiding:



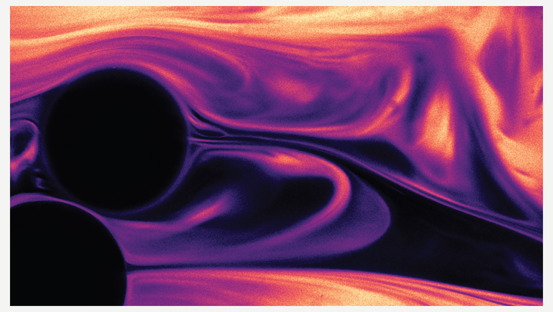
- Fundamental link between transverse dispersivity and Lyapunov exponent:

$$\lambda_\infty = \left(\frac{d}{Pe_T}\right)^{1/d}\langle\lambda_\sigma\rangle, \quad Pe_T \equiv \frac{\ell\langle v_1\rangle}{D_T^m}$$

- Different mechanism to pore scale.flow, larger Péclet number, up to $Pe \sim 10^5$
- Chaotic mixing arises even for weakly anisotropic and heterogeneous media



- Strong mixing for strongly heterogeneous media: $\lambda_\infty \xrightarrow{\sigma^2_{\ln k}} \ln 2$
- Mixing significantly impacts many processes: solute mixing, dispersion, reactions, colloids, etc:



- Chaotic mixing dynamics must be accounted for to predict and understand flow and transport at the Darcy scale

Thank you! Questions?