

# AN EFFICIENT METHOD OF DETERMINATION OF THE KLINKENBERG CORRECTION FOR SLIP FLOW IN POROUS MEDIA

D. LASSEUX<sup>1</sup>, T. ZAOUTER<sup>2</sup> & F. J. VALDÉS-PARADA<sup>3</sup>

1. I2M, UMR 5295, CNRS, Univ. Bordeaux, 351, Cours de la Libération, 33405 Talence CEDEX, France

2. CEA, DES, ISEC, DPME, SEME, Laboratoire d'Étanchéité, Univ. Montpellier, Marcoule, France

3. División de Ciencias Básicas e Ingeniería, Universidad Autónoma Metropolitana-Iztapalapa,  
Av. Ferrocarril San Rafael Atlixco 186, Col. Leyes de Reforma, 09310, CDMX, Mexico

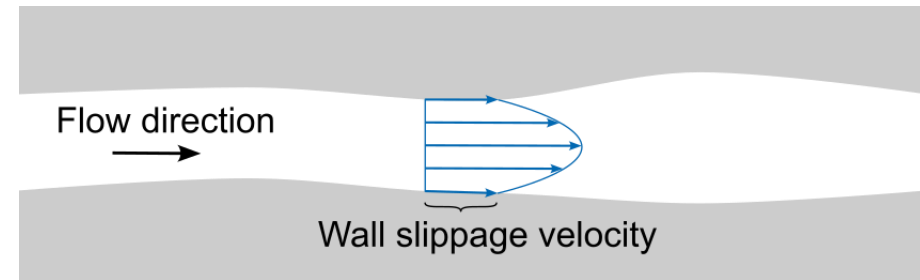
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# Rarefied gas flow in the slip regime

Slip flow is encountered, for instance

- For non-wetting fluids
- As an effective boundary conditions for flow over rough surfaces
- In rarefied gas flow, when KNUDSEN effects become significant
  - Geological gas storage & extraction, sealing, composite manufacturing...



Rarefied gas flow characterised by the KNUDSEN number

$$\text{Kn} = \frac{\lambda}{\ell}$$

← Gas mean free path ( $\propto 1/\rho$ )  
← Characteristic size (pore size)

For  $\text{Kn} \leq \mathcal{O}(10^{-1})$ , the flow is in the slip regime

- Flow modelled using NAVIER-STOKES equations with first-order slip boundary condition at the walls
  - $\mathcal{O}(\text{Kn})$  approximation to the BOLTZMANN equation

# One-phase microscopic slip flow model

Pore-scale microscopic flow model (slightly compressible, isothermal, creeping flow)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{in } \mathcal{V}_\beta \quad (1.1)$$

$$-\nabla p + \mu \nabla^2 \mathbf{v} = 0 \quad \text{in } \mathcal{V}_\beta \quad (1.2)$$

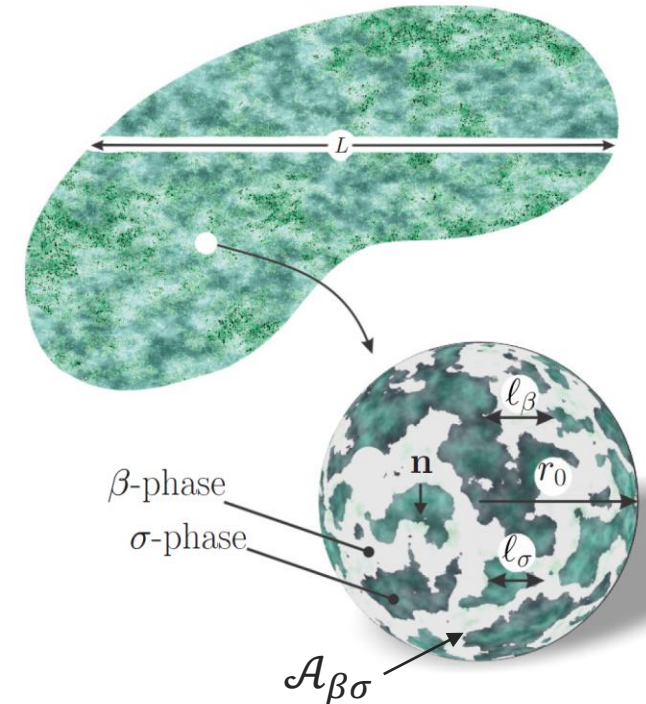
$$\rho = \varphi(p) \quad \text{in } \mathcal{V}_\beta \quad (1.3)$$

Slip BC  $\rightarrow$

$$\mathbf{v} = -\xi \lambda \mathbf{P} \cdot (\mathbf{n} \cdot (\nabla \mathbf{v} + \nabla \mathbf{v}^T)) \quad \text{on } \mathcal{A}_{\beta\sigma} \quad (1.4)$$

with

- $\mathbf{P} = \mathbf{I} - \mathbf{nn}$  the tangential projection tensor ( $\mathbf{P} = \mathbf{P}^T \geq 0$ ,  $\mathbf{P} = \mathbf{P} \cdot \mathbf{P}$ )
- $\xi$  the accommodation coefficient of the gas at the walls
- $\xi \lambda$  is identified as local slip length
  - For other slip mechanisms (non wetting flow, effective BC over rough surface): generally replaced by a slip length  $l$ .



# Macroscopic flow model with slip

Definition of averaging operators

$$\langle \cdot \rangle^\beta = \frac{1}{V_\beta} \int_{\mathcal{V}_\beta} \cdot \, dV$$

$$\langle \cdot \rangle = \frac{1}{V} \int \cdot \, dV = \varepsilon \langle \cdot \rangle^\beta$$

Porosity

After upscaling, considering slightly compressible flow ( $\rho \approx \langle \rho \rangle^\beta$ ,  $\lambda \approx \bar{\lambda} \propto 1/\langle \rho \rangle^\beta$ ), we obtain

Macroscopic model

$$\varepsilon \frac{\partial \langle \rho \rangle^\beta}{\partial t} + \nabla \cdot (\langle \rho \rangle^\beta \langle \mathbf{v} \rangle) = 0 \quad (2.1)$$

$$\langle \mathbf{v} \rangle = -\frac{\mathbf{K}}{\mu} \cdot \nabla \langle p \rangle^\beta \quad (2.2)$$

$$\langle \rho \rangle^\beta = \varphi(\langle p \rangle^\beta) \quad (2.3)$$

Unit cell closure problem

$$\nabla \cdot \mathbf{D} = \mathbf{0} \quad \text{in } \mathcal{V}_\beta \quad (3.1)$$

$$\nabla \cdot \mathbf{S} + \mathbf{I} = \mathbf{0} \quad \text{in } \mathcal{V}_\beta \quad (3.2)$$

$$\mathbf{S} = -\mathbf{Id} + \nabla \mathbf{D} + \nabla \mathbf{D}^{\text{T}12} \quad \text{in } \mathcal{V}_\beta \quad (3.3)$$

$$\mathbf{D} = -\xi \bar{\lambda} \mathbf{P} \cdot (\mathbf{n} \cdot \mathbf{S}) \quad \text{on } \mathcal{A}_{\beta\sigma} \quad (3.4)$$

$$\mathbf{d}, \mathbf{D} \text{ periodic, } \langle \mathbf{d} \rangle = \mathbf{0} \quad (3.5)$$

$$\mathbf{K} = \langle \mathbf{D} \rangle \quad \text{Dependence on } \xi \bar{\lambda} \quad (3.6)$$

Apparent permeability tensor

It can be shown that  $\mathbf{K}$  is symmetric and positive definite.

# Viscous and slip contributions

To isolate purely viscous effects from slip effects, power series in  $\xi\bar{\lambda}$  are proposed as

$$\mathbf{d} = \sum_{j=0}^{+\infty} \mathbf{d}_j (\xi\bar{\lambda})^j \quad \mathbf{D} = \sum_{j=0}^{+\infty} \mathbf{D}_j (\xi\bar{\lambda})^j \quad \mathbf{K} = \sum_{j=0}^{+\infty} \mathbf{K}_j (\xi\bar{\lambda})^j = \mathbf{K}_0 + \xi\bar{\lambda}\mathbf{K}_1 + (\xi\bar{\lambda})^2 \mathbf{K}_2 + \dots$$

Identification at the different orders leads to sequentially coupled, intrinsic, closure problems

For  $j = 0$

$$\nabla \cdot \mathbf{D}_0 = \mathbf{0} \quad \text{in } \mathcal{V}_\beta \quad (4.1)$$

$$\nabla \cdot \mathbf{S}_0 + \mathbf{I} = \mathbf{0} \quad \text{in } \mathcal{V}_\beta \quad (4.2)$$

$$\mathbf{S}_0 = -\mathbf{I}d_0 + \nabla\mathbf{D}_0 + \nabla\mathbf{D}_0^{\text{T}12} \quad \text{in } \mathcal{V}_\beta \quad (4.3)$$

$$\mathbf{D}_0 = \mathbf{0} \quad \text{on } \mathcal{A}_{\beta\sigma} \quad (4.4)$$

$$d_0, \mathbf{D}_0 \text{ periodic, } \langle d_0 \rangle = \mathbf{0} \quad (4.5)$$

$$\mathbf{K}_0 = \langle \mathbf{D}_0 \rangle \quad (4.6)$$

For  $j \geq 1$

$$\nabla \cdot \mathbf{D}_j = \mathbf{0} \quad \text{in } \mathcal{V}_\beta \quad (5.1)$$

$$\nabla \cdot \mathbf{S}_j = \mathbf{0} \quad \text{in } \mathcal{V}_\beta \quad (5.2)$$

$$\mathbf{S}_j = -\mathbf{I}d_j + \nabla\mathbf{D}_j + \nabla\mathbf{D}_j^{\text{T}12} \quad \text{in } \mathcal{V}_\beta \quad (5.3)$$

$$\mathbf{D}_j = -\mathbf{P} \cdot (\mathbf{n} \cdot \mathbf{S}_{j-1}) \quad \text{on } \mathcal{A}_{\beta\sigma} \quad (5.4)$$

$$d_j, \mathbf{D}_j \text{ periodic, } \langle d_j \rangle = \mathbf{0} \quad (5.5)$$

$$\mathbf{K}_j = \langle \mathbf{D}_j \rangle \quad (5.6)$$

Coupling

Intrinsic permeability tensor (no-slip)

$j^{\text{th}}$  order slip correction tensor

# Alternative expressions of the correction tensors

Up to now, we have sequentially coupled intrinsic closure problems

- To obtain the  $j^{\text{th}}$  order correction: need to solve for  $j + 1$  problems
- Potentially time consuming
- Prone to propagation of numerical errors

☞ Alternative expressions for  $\mathbf{K}_j$  are hoped for

Considering the following GREEN reciprocity formula

$$\int_{\mathcal{V}_\beta} \mathbf{D}_p^T \cdot (\nabla \cdot \mathbf{S}_q) - (\nabla \cdot \mathbf{S}_p)^T \cdot \mathbf{D}_q \, dV = \int_{\mathcal{A}_{\beta\sigma}} \mathbf{D}_p^T \cdot (\mathbf{n} \cdot \mathbf{S}_q) - (\mathbf{n} \cdot \mathbf{S}_p)^T \cdot \mathbf{D}_q \, dA$$

Letting  $p = 0$  and  $q = j$ , we obtain

$$\mathbf{K}_j = -\frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} (\mathbf{n} \cdot \mathbf{S}_0)^T \cdot \mathbf{D}_j \, dA = \frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} \underbrace{(\mathbf{n} \cdot \mathbf{S}_0)^T \cdot \mathbf{P}}_{\text{Using slip BC (5.4)}} \cdot \underbrace{(\mathbf{n} \cdot \mathbf{S}_{j-1})}_{\text{Using transpose of slip BC (5.4)}} \, dA = -\frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} \mathbf{D}_1^T \cdot (\mathbf{n} \cdot \mathbf{S}_{j-1}) \, dA$$

# Alternative expressions of the correction tensors

For  $p, q \neq 0$ , the volume integral in GREEN's formula is identically zero ( $\mathbf{S}_j$  are solenoidal), giving

$$\int_{\mathcal{A}_{\beta\sigma}} \mathbf{D}_p^T \cdot (\mathbf{n} \cdot \mathbf{S}_q) dA = \int_{\mathcal{A}_{\beta\sigma}} (\mathbf{n} \cdot \mathbf{S}_p)^T \cdot \mathbf{D}_q dA$$

Using  $p = 1$  et  $q = j - 1$ , we obtain

$$\mathbf{K}_j = -\frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} (\mathbf{n} \cdot \mathbf{S}_0)^T \cdot \mathbf{D}_j dA = -\frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} \mathbf{D}_1^T \cdot (\mathbf{n} \cdot \mathbf{S}_{j-1}) dA = -\frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} (\mathbf{n} \cdot \mathbf{S}_1)^T \cdot \mathbf{D}_{j-1} dA$$

Repeating the process recursively until  $p = q = \lfloor j/2 \rfloor$ , we finally obtain

$$\mathbf{K}_j = \begin{cases} -\frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} (\mathbf{n} \cdot \mathbf{S}_{j/2})^T \cdot \mathbf{D}_{j/2} dA & j \text{ even} \\ \frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} (\mathbf{n} \cdot \mathbf{S}_{(j-1)/2})^T \cdot \mathbf{P} \cdot (\mathbf{n} \cdot \mathbf{S}_{(j-1)/2}) dA & j \text{ odd} \end{cases} \quad (6)$$

# Alternative expressions of the correction tensors

This gives for the first few tensors  $\mathbf{K}_j$

$$\mathbf{K}_0 = \frac{1}{V} \int_{\mathcal{V}_\beta} \mathbf{D}_0 \, dV$$

$$\mathbf{K}_1 = \frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} (\mathbf{n} \cdot \mathbf{S}_0)^T \cdot \mathbf{P} \cdot (\mathbf{n} \cdot \mathbf{S}_0) \, dA$$

} Requires only solution to 0<sup>th</sup> order closure

$$\mathbf{K}_2 = -\frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} (\mathbf{n} \cdot \mathbf{S}_1)^T \cdot \mathbf{D}_1 \, dA$$

$$\mathbf{K}_3 = \frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} (\mathbf{n} \cdot \mathbf{S}_1)^T \cdot \mathbf{P} \cdot (\mathbf{n} \cdot \mathbf{S}_1) \, dA$$

} Requires only solution to 1<sup>st</sup> order closure

Each closure problem now provides 2 slip correction tensors!

- Computational expense divided by 2 to reach a given correction order

Specifically, KLINKENBERG's correction (1<sup>st</sup> order truncation,  $\mathbf{K} \approx \mathbf{K}_0 + \xi \bar{\lambda} \mathbf{K}_1$ ) can be obtained by solving the no-slip 0<sup>th</sup> order closure only!

# Properties of the correction tensors

We have the following equivalent expressions for  $j \geq 1$

$$\mathbf{K}_j = \begin{cases} -\frac{1}{V} \int_{\mathcal{V}_\beta} \frac{1}{2} (\nabla \mathbf{D}_{j/2} + \nabla \mathbf{D}_{j/2}^{\text{T}_{12}})^{\text{T}_{13}} : (\nabla \mathbf{D}_{j/2} + \nabla \mathbf{D}_{j/2}^{\text{T}_{12}}) dV & j \text{ even} & \leftarrow \text{Integrand of the form } \mathbf{A}^{\text{T}_{13}} : \mathbf{A} \\ \frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} (\mathbf{n} \cdot \mathbf{S}_{(j-1)/2})^{\text{T}} \cdot \mathbf{P} \cdot (\mathbf{n} \cdot \mathbf{S}_{(j-1)/2}) dA & j \text{ odd} & \leftarrow \text{Integrand of the form } \mathbf{A}^{\text{T}} \cdot \mathbf{A} \end{cases} \quad (7)$$

For the first few tensors, we have

$$\mathbf{K}_0 = +\frac{1}{V} \int_{\mathcal{V}_\beta} \frac{1}{2} (\nabla \mathbf{D}_0 + \nabla \mathbf{D}_0^{\text{T}_{12}})^{\text{T}_{13}} : (\nabla \mathbf{D}_0 + \nabla \mathbf{D}_0^{\text{T}_{12}}) dV$$

$$\mathbf{K}_1 = +\frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} (\mathbf{n} \cdot \mathbf{S}_0)^{\text{T}} \cdot \mathbf{P} \cdot (\mathbf{n} \cdot \mathbf{S}_0) dA$$

$$\mathbf{K}_2 = -\frac{1}{V} \int_{\mathcal{V}_\beta} \frac{1}{2} (\nabla \mathbf{D}_1 + \nabla \mathbf{D}_1^{\text{T}_{12}})^{\text{T}_{13}} : (\nabla \mathbf{D}_1 + \nabla \mathbf{D}_1^{\text{T}_{12}}) dV$$

$$\mathbf{K}_3 = +\frac{1}{V} \int_{\mathcal{A}_{\beta\sigma}} (\mathbf{n} \cdot \mathbf{S}_1)^{\text{T}} \cdot \mathbf{P} \cdot (\mathbf{n} \cdot \mathbf{S}_1) dA$$

Which proves that

- All  $\mathbf{K}_j$  are symmetric
- and form an alternate series for  $j \geq 1$  ( $\mathbf{K}_j \leq 0$  for  $j$  even,  $\mathbf{K}_j \geq 0$  for  $j$  odd)

# PADÉ approximant

Having the power series for  $\mathbf{K}$  at hand, we propose to construct the PADÉ approximant,  $\tilde{\mathbf{K}}_{(1,1)}$

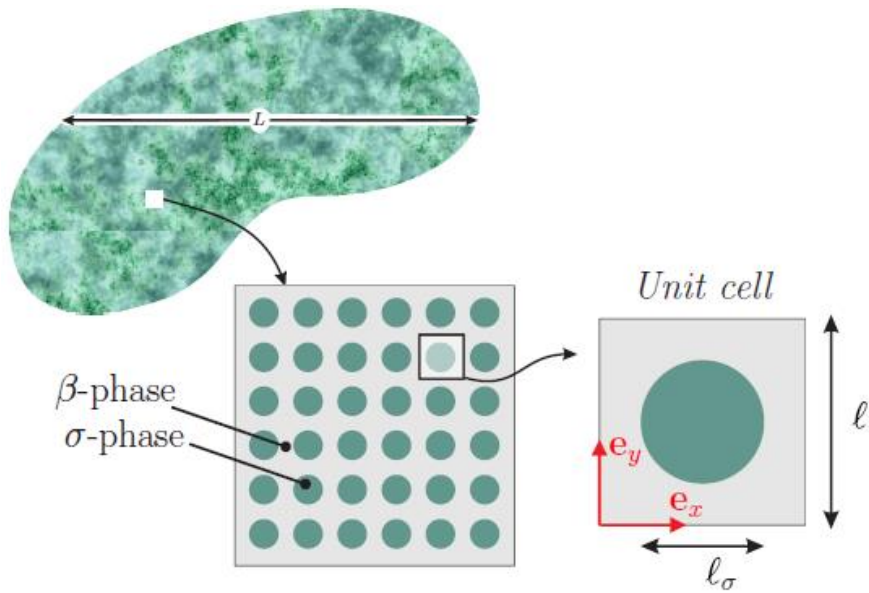
- From the solutions of the first 2 closure problems only (providing  $\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ )
- Same expansion as the power series up to order 2 in  $\xi\bar{\lambda}$
- Constant asymptotic behaviour as  $\xi\bar{\lambda} \rightarrow +\infty$  (perfect slip) [Lasseux et al., J. Fluid Mech. 997, A65 \(2024\)](#)
- Better extrapolation properties than a polynomial

$$\tilde{\mathbf{K}}_{(1,1)} = \underbrace{\left( \mathbf{K}_0 + \xi\bar{\lambda}(\mathbf{K}_1 - \mathbf{K}_0 \cdot \mathbf{K}_1^{-1} \cdot \mathbf{K}_2) \right)}_{\text{“Numerator”}} \cdot \underbrace{\left( \mathbf{I} - \xi\bar{\lambda}\mathbf{K}_1^{-1} \cdot \mathbf{K}_2 \right)^{-1}}_{\text{“Denominator”}}$$

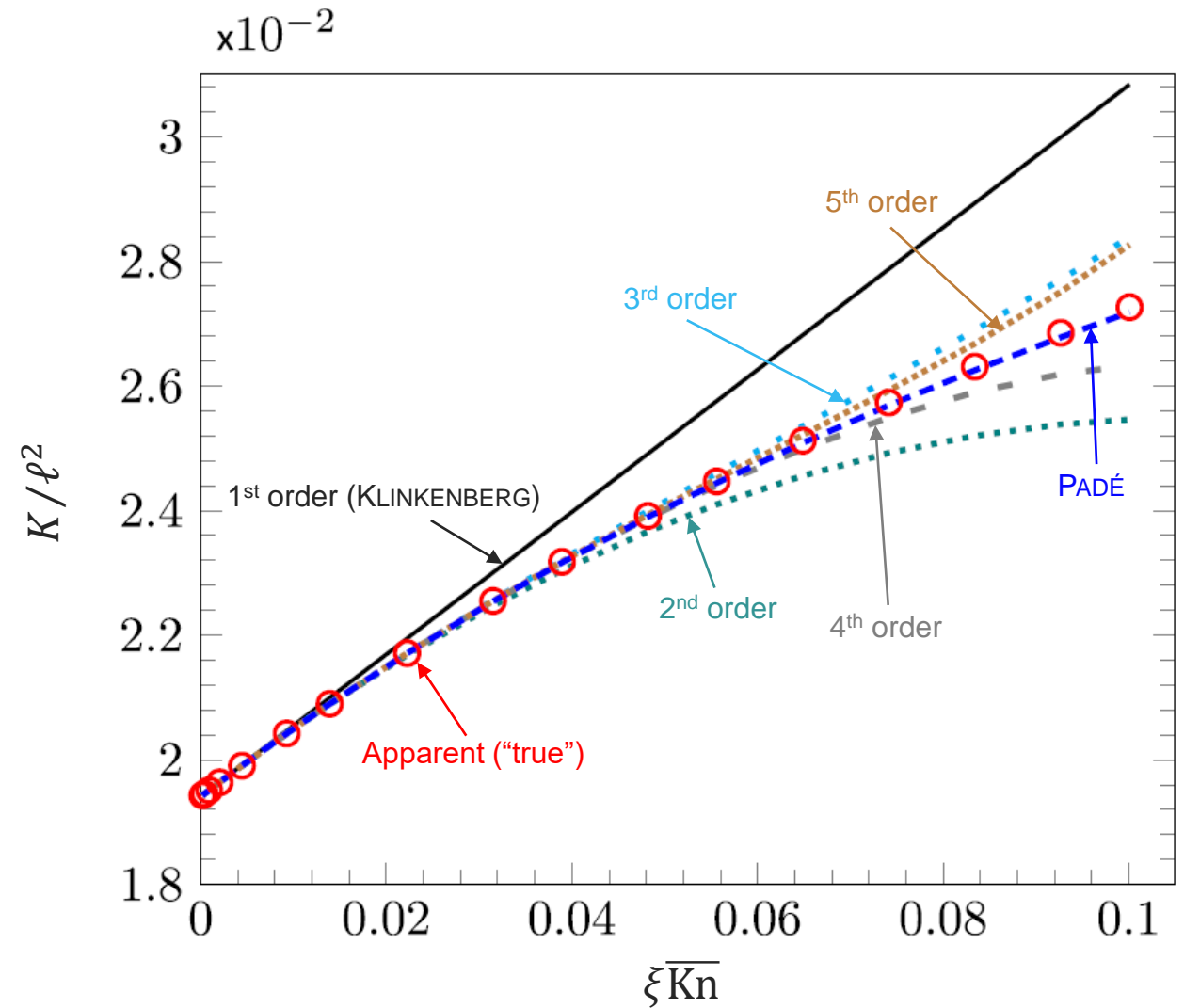
$\tilde{\mathbf{K}}_{(1,1)}$  is symmetric

# Numerical example

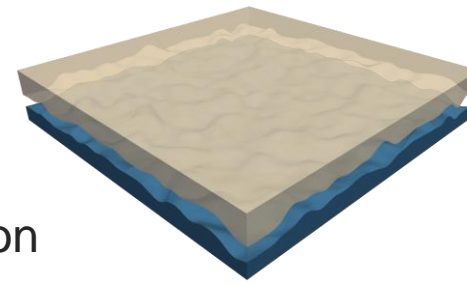
Consider the bundle of cylinders



$$\varepsilon = 0.8, \quad \overline{Kn} = \frac{\bar{\lambda}}{l_\beta}, \quad l_\beta = \sqrt{\frac{12K_0}{\varepsilon}}$$



# Case of a rough fracture



After upscaling, we obtain the macroscopic 2D REYNOLDS equation

2D porosity

Macroscopic model

Unit cell closure problem

$$\varepsilon_\beta \langle h \rangle \frac{\partial \langle \rho \rangle^\beta}{\partial t} + \nabla \cdot \langle \mathbf{q} \rangle = 0 \quad (8.1)$$

$$\langle \mathbf{q} \rangle = -\langle \rho \rangle^\beta \frac{\mathbf{K}}{\mu} \cdot \nabla \langle p \rangle^\beta \quad (8.2)$$

$$\langle \rho \rangle^\beta = \varphi(\langle \rho \rangle^\beta) \quad (8.3)$$

$$\nabla \cdot \mathbf{F} = \mathbf{0} \quad \text{in } \mathcal{A}_\beta \quad (9.1)$$

$$\mathbf{F} = \bar{k}(\mathbf{I} + \nabla \mathbf{b}) \quad \text{in } \mathcal{A}_\beta \quad (9.2)$$

$$\bar{k} = k_0 + 6\xi \bar{\lambda} k_1 \quad \text{in } \mathcal{A}_\beta \quad (9.3)$$

$$\mathbf{n} \cdot \mathbf{F} = \mathbf{0} \quad \text{on } \mathcal{C}_{\beta\sigma} \quad (9.4)$$

$$\mathbf{b} \text{ periodic, } \langle \mathbf{b} \rangle = \mathbf{0} \quad (9.5)$$

$$\mathbf{K} = \langle \mathbf{F} \rangle \quad (9.6)$$

Apparent transmissivity tensor

With  $\mathbf{b} = \mathbf{b}_0 + 6\xi \bar{\lambda} \mathbf{b}_1 + (6\xi \bar{\lambda})^2 \mathbf{b}_2 + \dots$

$\mathbf{K} = \mathbf{K}_0 + 6\xi \bar{\lambda} \mathbf{K}_1 + (6\xi \bar{\lambda})^2 \mathbf{K}_2 + \dots$

We are able to obtain

$$\mathbf{K}_0 = \langle k_0 (\mathbf{I} + \nabla \mathbf{b}_0)^T \cdot (\mathbf{I} + \nabla \mathbf{b}_0) \rangle$$

$$\mathbf{K}_1 = \langle k_1 (\mathbf{I} + \nabla \mathbf{b}_0)^T \cdot (\mathbf{I} + \nabla \mathbf{b}_0) \rangle$$

$$\mathbf{K}_j = \begin{cases} -\langle k_0 \nabla \mathbf{b}_{j/2}^T \cdot \nabla \mathbf{b}_{j/2} \rangle, & j \text{ even} \\ \langle k_1 \nabla \mathbf{b}_{(j-1)/2}^T \cdot \nabla \mathbf{b}_{(j-1)/2} \rangle, & j \text{ odd} \end{cases}, \quad j \geq 2$$

(almost) perfect analogy  
with 3D porous media case

# Conclusion

- Each closure problem gives access to 2 slip correction tensors
- Solving the first  $n$  closure problems provide  $2n$  permeability and slip correction tensors  $(\mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{2n-1})$
- KLINKENBERG's model can be obtained from the intrinsic no-slip 0<sup>th</sup> order closure problem only!
- All tensors are symmetric and alternatively positive and negative definite, forming an alternate series
- PADÉ approximant can be built upon the power series to better capture the dependence of  $\mathbf{K}$  with  $\xi \overline{K_n}$

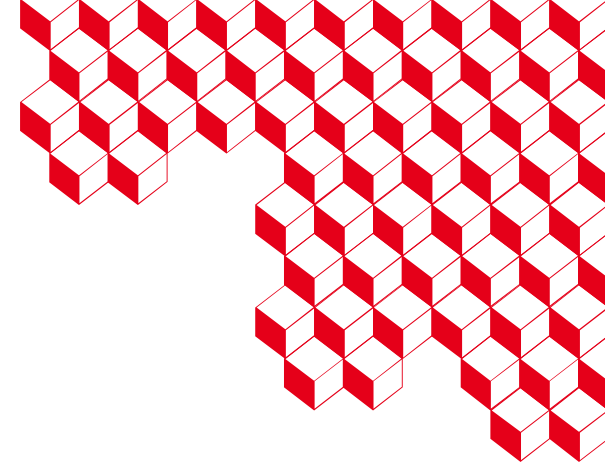
# References

## Porous case

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## Fracture case

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# Thank you for your attention!

Tony ZAOUTER

DES/ISEC/DPME/SEME/LE

✉ : Laboratoire d'Étanchéité  
2, rue James WATT  
26 700 Pierrelatte

☎ : 04 75 27 39 15

✉ : [tony.zaouter@cea.fr](mailto:tony.zaouter@cea.fr)