Resistance-Distance-Based Coarse Graining of Flow Networks via Gradient-Based Conductivity Estimation

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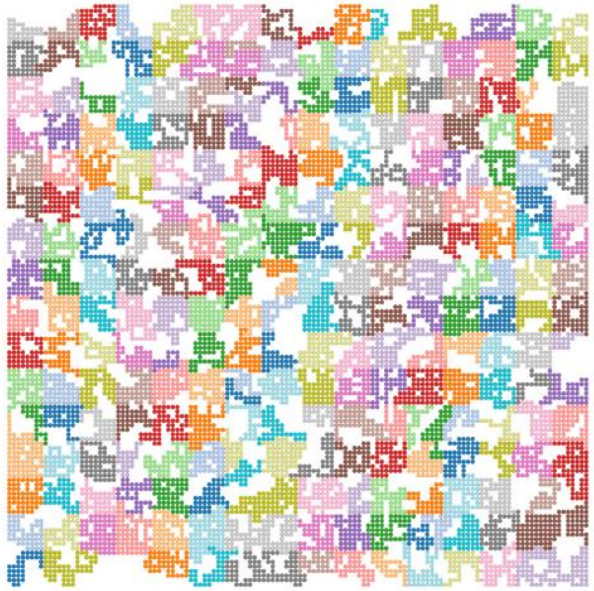
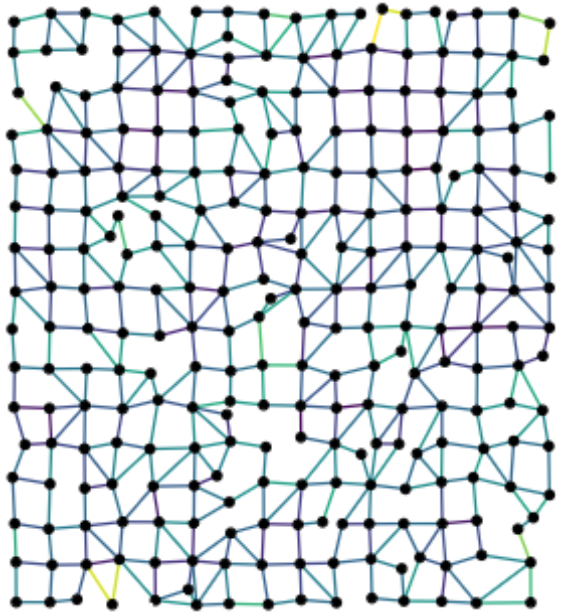
IFP Energies Nouvelles (IFPEN)





Context

Karst systems : Very heterogeneous natural underground pipes at multiple scales

ERC Synergy Karst project: predict flow and transport in complex karstic systems in regard with extreme rain or drought events

WP1 Conduit scale

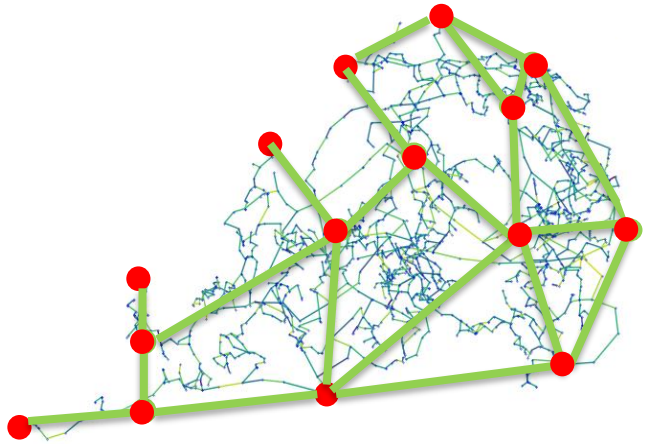
WP2 Constructing the network

WP3 Flow & transport net scale

WP4 Case studies and Speleogenesis

Motivation

- Large flow networks are computationally demanding to simulate in detail.
- Simplifying the network topology reduces cost but may misrepresent how flow is distributed through the system.
- The main challenge is to **preserve the hydraulic response** after coarsening.



- This requires defining **effective conductances** on the coarse edges that are **physically consistent** with the fine-scale structure.

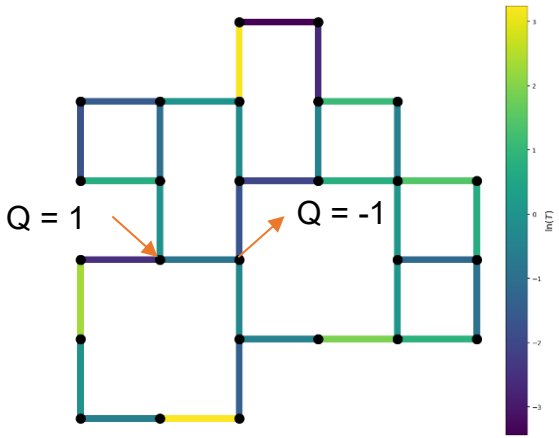
Global Upscaling of Flow Networks

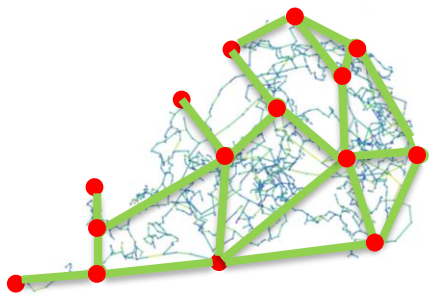
Develop a framework for global upscaling of heterogeneous flow networks.

Unlike local averaging approaches, global upscaling aims to retain the influence of the entire network on coarse-scale flow behavior

Edge Resistance Distance

$$Q_{ij} = T_{ij}(P_j - P_i)$$





For unit flow:

$$R_{ij} = P_j - P_i$$

$$R_{ij} \neq 1/T_{ij}$$

Recover effective coarse-scale conductance consistent with resistance responses.

Computation of conductances at the coarse network

Find coarse conductances $\mathbf{T}$ such that the coarse network reproduces target resistance responses.

$$\Phi(T) = \sum_{\text{edges } ij} \left(R_{ij}^{\text{given}} - R_{ij}(\{T_{kl}\}) \right)^2$$

Gauss–Newton optimization

conductance update:

$$\mathbf{J}^\top \mathbf{J} \cdot \delta \mathbf{T} = -\mathbf{J}^\top \mathbf{r}$$

$$T^{n+1} = T^n + \delta T$$

$$r_{ij} = R_{ij}(T_{kl}) - R_{ij}^{\text{given}}$$

with Jacobian

$$J_{ij,kl} = \frac{\partial R_{ij}}{\partial T_{kl}}$$

To solve this nonlinear inverse problem, we use a Gauss–Newton iterative scheme. At each iteration, the conductances T_{kl} are updated from a linearization of the resistance responses.

Derivation of sensitivities

Resistance distance as an energy minimization

For a unit dipole between nodes i and j, define the energy functional

$$E_{ij}(\mathbf{P}) = \frac{1}{2}\mathbf{P}^{\top} \cdot \mathbf{L} \cdot \mathbf{P} - (P_i - P_j)$$

The physical dipole solution minimizes this functional:

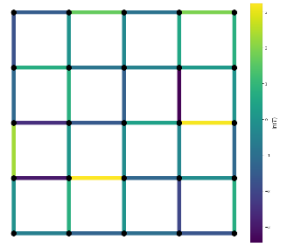
$$\min_{\mathbf{P}} E_{ij}(\mathbf{P}) = -\frac{1}{2}R_{ij}$$

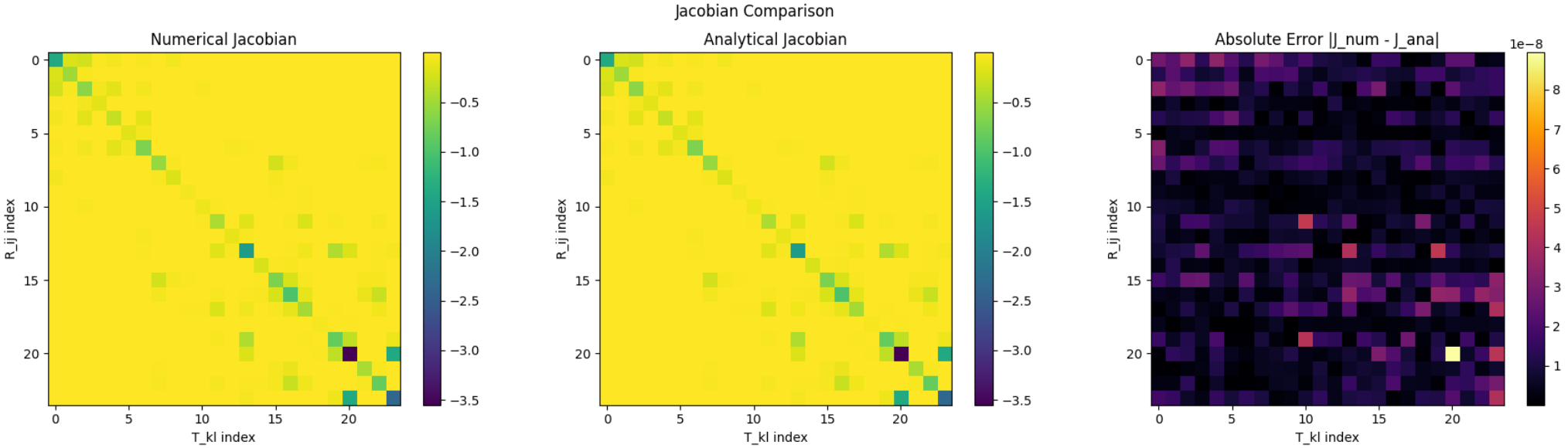
Using the variational formulation:

$$J_{ij,kl} = \frac{\partial R_{ij}}{\partial T_{kl}} = -\left(P^*_{ij,k} - P^*_{ij,l}\right)^2$$

Derivatives are obtained during the computation of Rij without any additional cost

Jacobian Comparison: Numerical vs Analytical

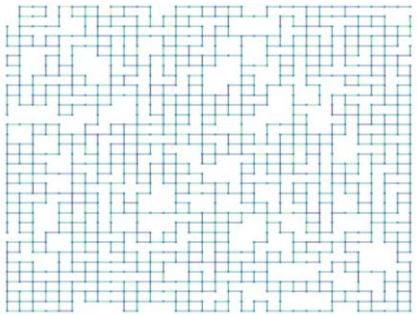


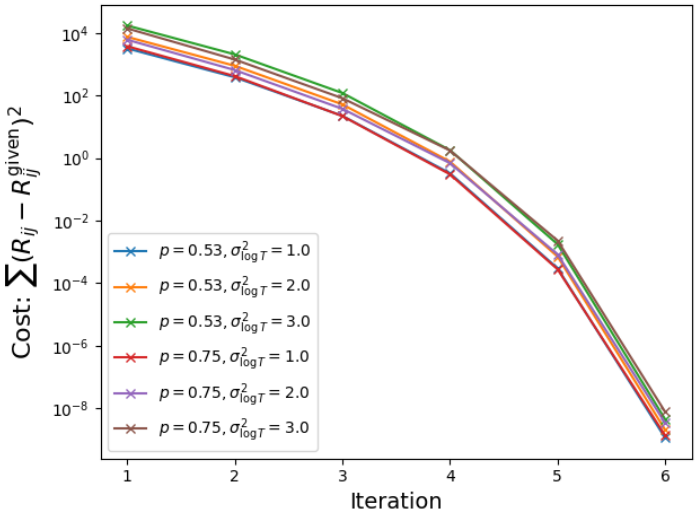


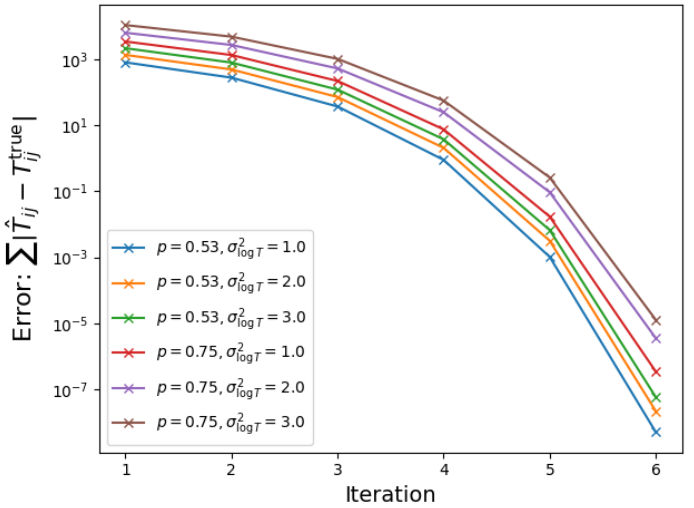
These results were obtained on a small percolation network generated with edge T_{ij} drawn from a lognormal distribution of geometric mean 1 and logarithmic variance 1, with occupation probability $p = 1$ on a domain of size $N_x = N_y = 4$.

Conductances from Resistance distance: Validation

- Edge conductances T_{true} are drawn from a lognormal distribution, on percolation networks
- Using these values, the corresponding resistance distances R_{given} are computed
- The Gauss-Newton algorithm is then used to iteratively minimize the cost function.
- Finally, we verify that the recovered conductances T_{ij} match the original T_{true}







Conclusions

The method provides an **efficient way to recover effective conducances** from the coarse resistance structure through **Gauss–Newton optimization**.

Perspectives / Open questions

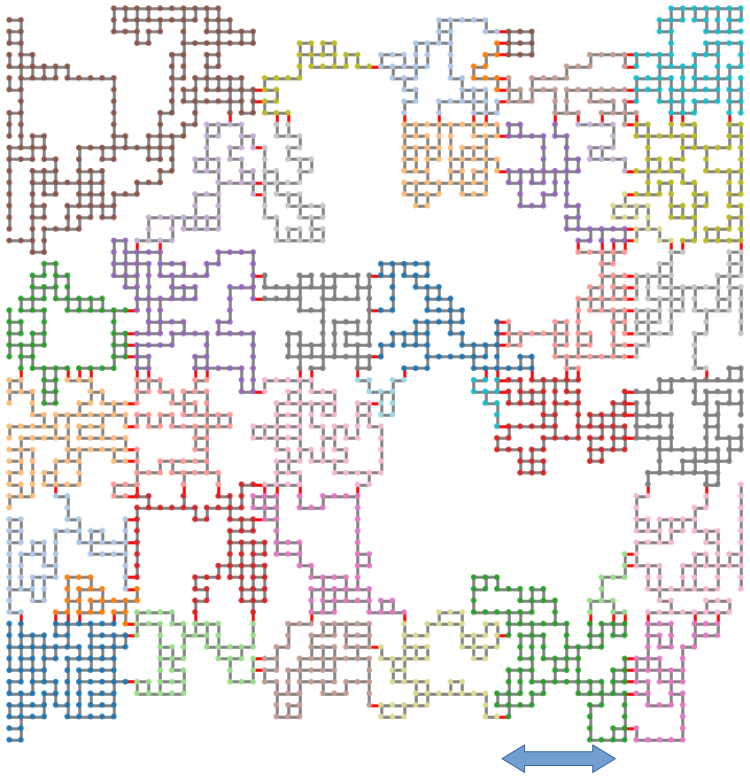
How can coarse regions and connections be defined to capture bottlenecks and long-range flow organization?

How to **quantitatively assess** whether the coarse network truly reproduces the fine-scale flow?

Thanks!

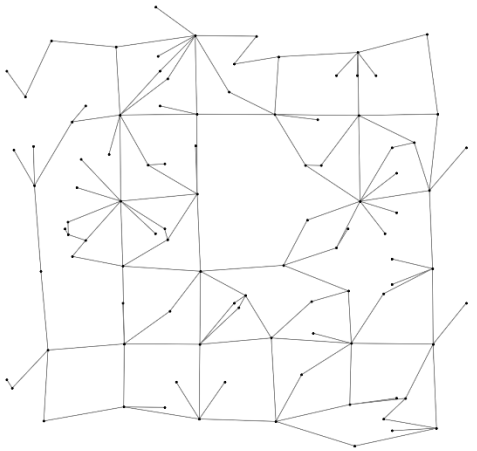
Coarsening procedure

- Start with Backbone Random Resistor Network
- Partitioning: define set of nodes to be grouped and edges in the coarse network



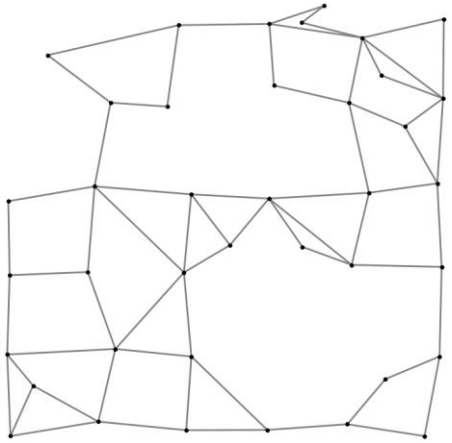
Geometrical: λ_c

Disconnected Coarse Network

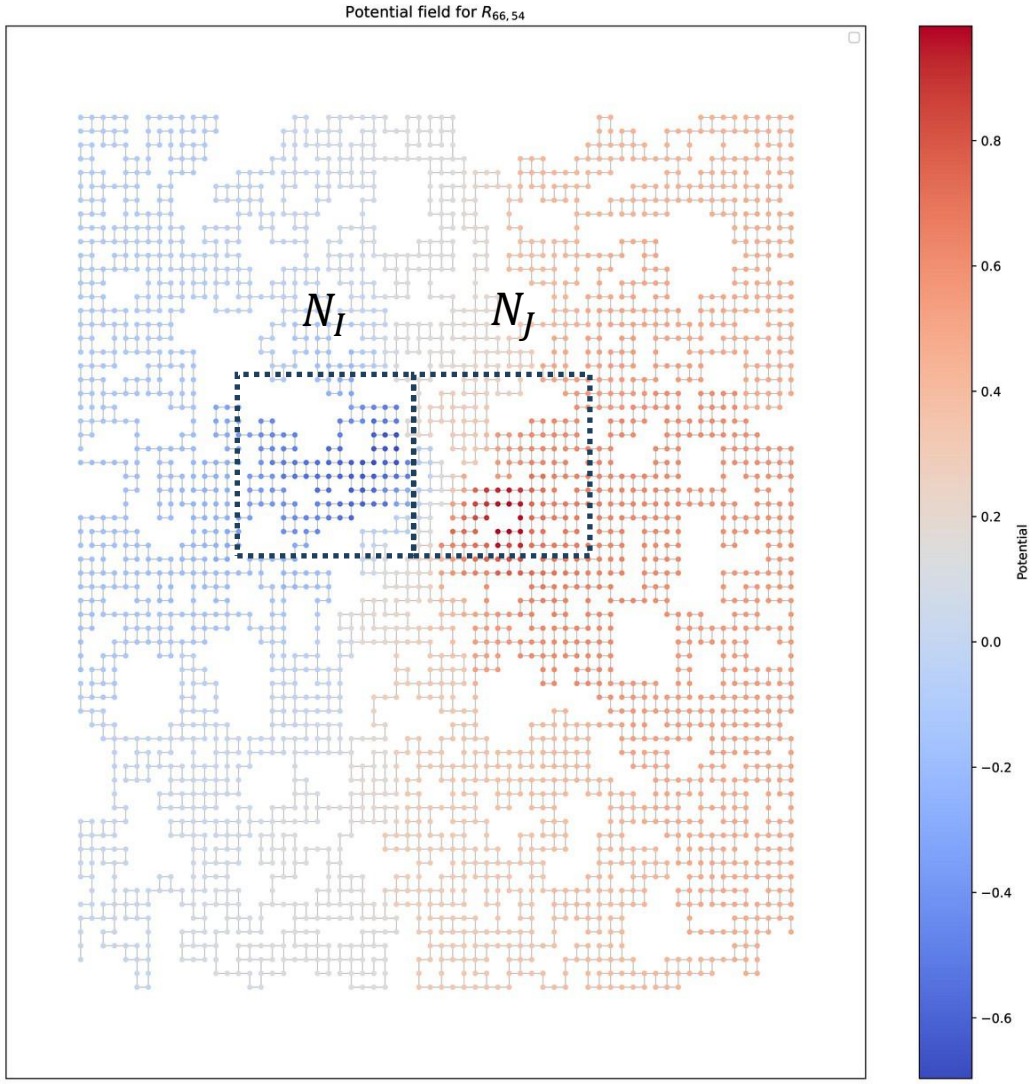


Connected Coarse Network

Does a physically consistent set of conductivities exist for any given resistance distance definition?



2) Computation of Effective Resistance Distances



- Option 1: Lagrange multipliers to avoid large potential gradients

$$R_{IJ} = \left(\frac{1}{N_I}\mathbf{1}_I - \frac{1}{N_J}\mathbf{1}_J\right)^\top \cdot \mathbf{L}^\dagger \cdot \left(\frac{1}{N_I}\mathbf{1}_I - \frac{1}{N_J}\mathbf{1}_J\right)$$

Inject an unit of flow at nodes in N_I **uniformly**

Extract an unit of flow at nodes in N_J **uniformly**

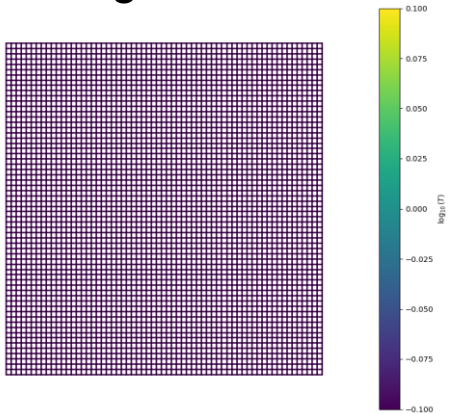
- Effective Graph Resistance

$$R_{\text{eff}}(A,B) = \frac{1}{|A||B|}\sum_{i\in A}\sum_{j\in B} R_{\text{eff}}(i,j)$$

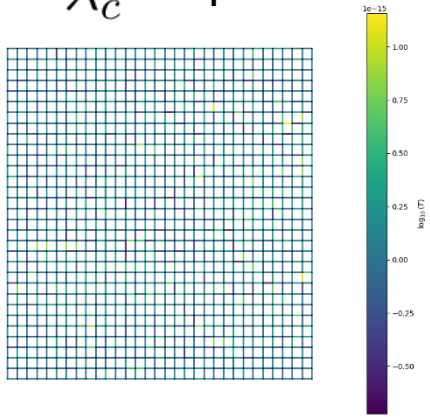
Compute the Resistance distance beween all pair of nodes I and J

Tests on Homogeneous Networks

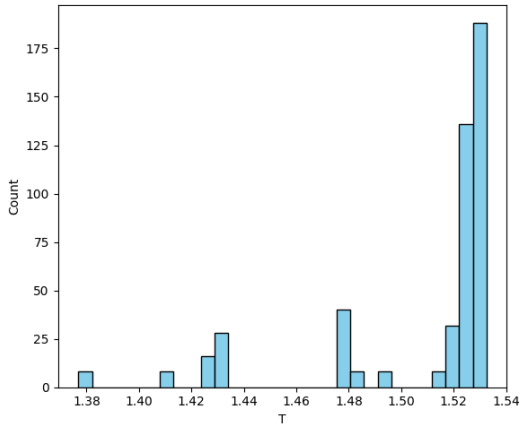
Fine scale
Homogeneus Network



λ_c = 1

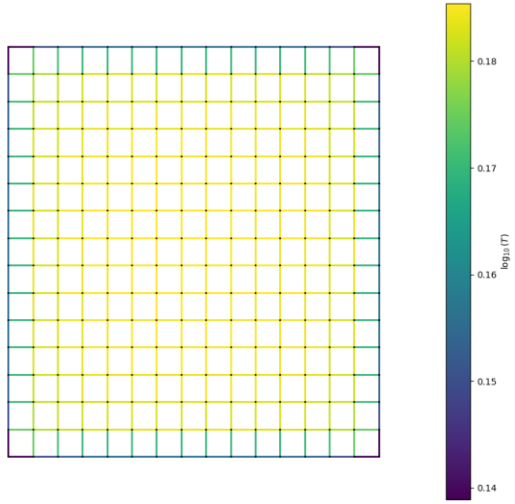


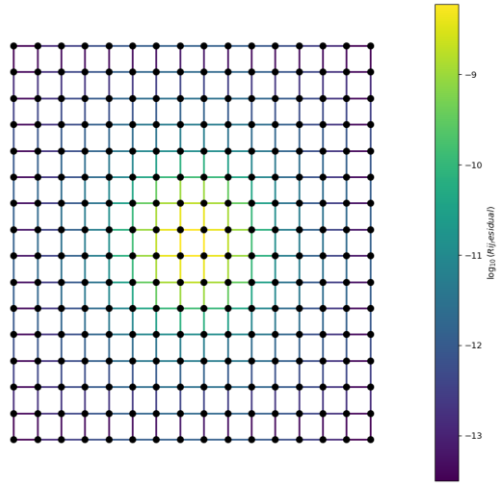
λ_c = 2

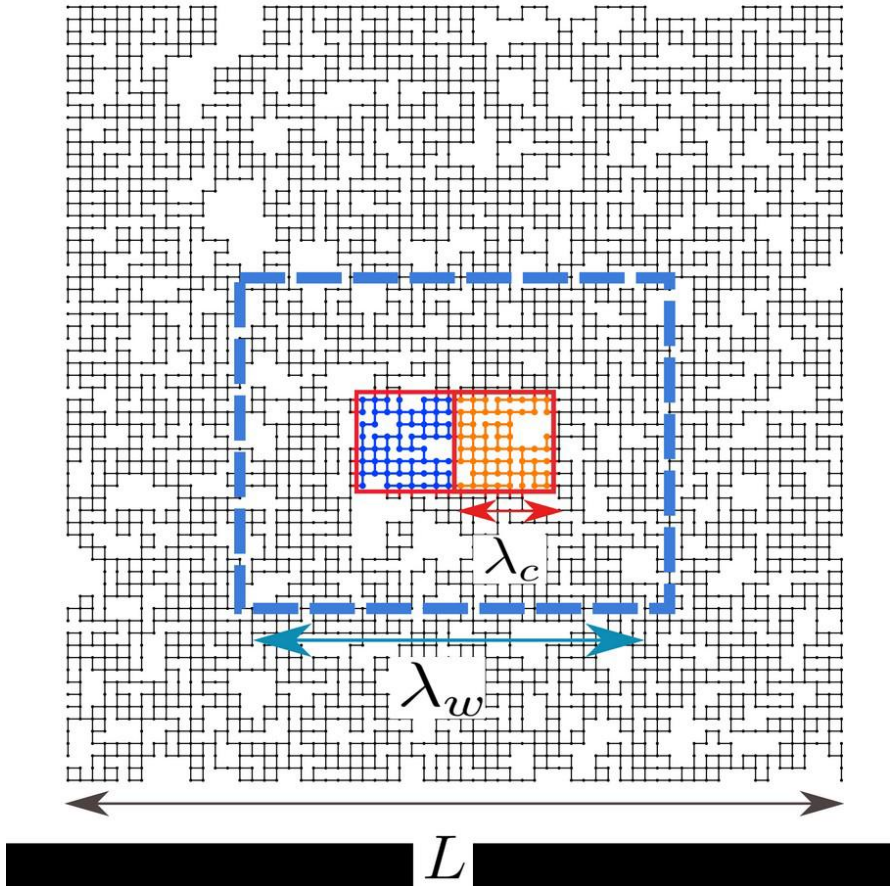


Homogeneus but
Conductivity T ≠ 1

λ_c = 2






λ_c
λ_w
L

A faithful coarse-graining should therefore preserve the low-frequency eigenmodes of the fine network

$$S_k = \frac{\mathbf{v}_k^{\text{fine}\to\text{coarse}} \cdot \mathbf{v}_k^{\text{coarse}}}{\|\mathbf{v}_k^{\text{fine}\to\text{coarse}}\| \, \|\mathbf{v}_k^{\text{coarse}}\|},$$