A Comparative Study of Fine-scale and Multi-scale Finite-Volume and Finite-Element Methods for Coupled Poroelastic Problmes

Mahsa Mehrazar
Cornelis Vuik
Mohammed Al Kobaisi
Hadi Hajibeygi

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Hydro-Mechanical Governing Equations

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - \alpha p \, \mathbf{I}$$

$$\boldsymbol{\sigma}' = \mathbf{C} : \boldsymbol{\varepsilon}$$

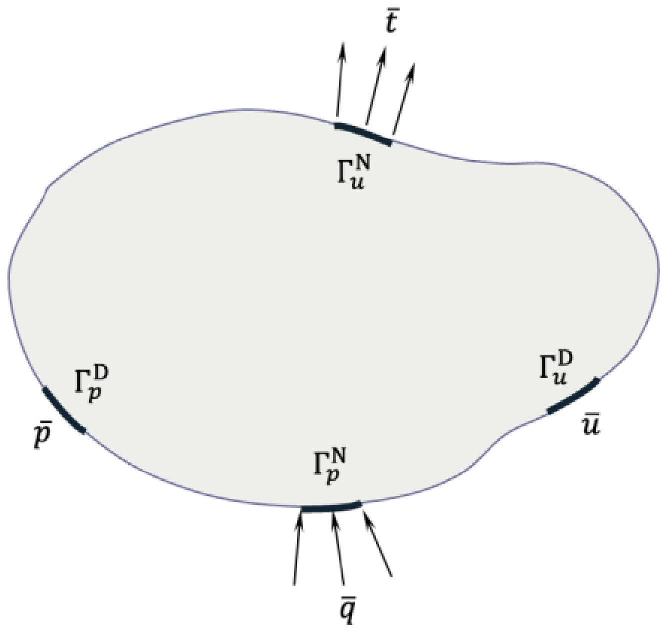
$$\boldsymbol{\varepsilon}(\mathbf{u}) = \nabla^s \mathbf{u} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$$

$$\mathbf{C} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

$$\alpha \frac{\partial \varepsilon_v}{\partial t} + \frac{1}{Q}\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{w} = q$$

$$\varepsilon_v = \mathrm{tr}(\boldsymbol{\varepsilon}) = \nabla \cdot \mathbf{u}$$

$$\mathbf{w} = -\boldsymbol{\Lambda} \nabla p \qquad \boldsymbol{\Lambda} = \frac{\mathbf{k}}{\mu}$$



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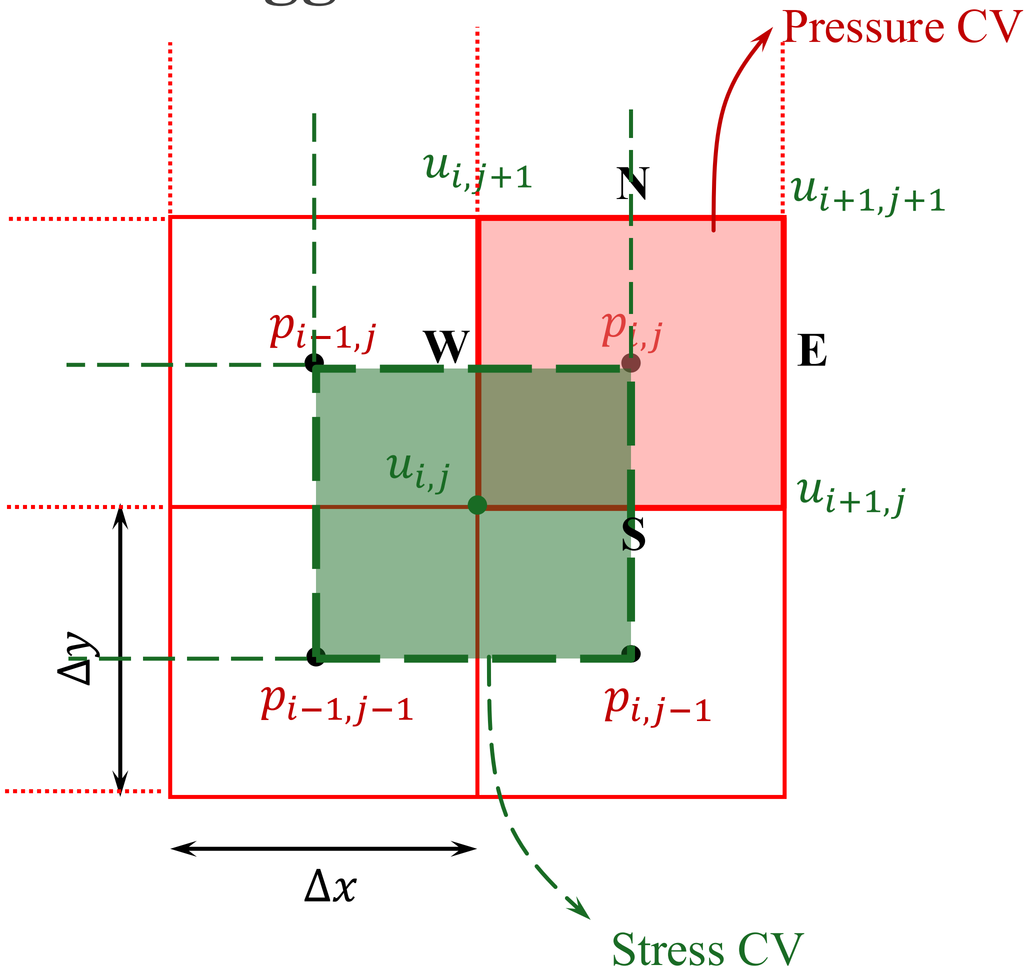
Linear System

$$\underbrace{\begin{bmatrix} \boldsymbol{K}_{uu} & -\boldsymbol{Q}_{up} \\ \frac{1}{\Delta t}\boldsymbol{Q}_{pu} & \boldsymbol{R}_{pp} + \frac{1}{\Delta t}\boldsymbol{S}_{pp} \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{Bmatrix} \boldsymbol{u}^{(n+1)} \\ \boldsymbol{p}^{(n+1)} \end{Bmatrix}}_{\mathbf{X}} = \underbrace{\begin{Bmatrix} \boldsymbol{F}_u \\ \boldsymbol{F}_p + \frac{1}{\Delta t}\left(\boldsymbol{S}_{pp}\boldsymbol{p}^{(n)} + \boldsymbol{Q}_{pu}\boldsymbol{u}^{(n)}\right) \end{Bmatrix}}_{\mathbf{F}}$$

$$\dot{\mathbf{X}}_{n+1} = \frac{\mathbf{X}_{n+1} - \mathbf{X}_n}{\Delta t}, \qquad \mathbf{X} = \begin{Bmatrix} \mathbf{U} \\ \mathbf{P} \end{Bmatrix}$$

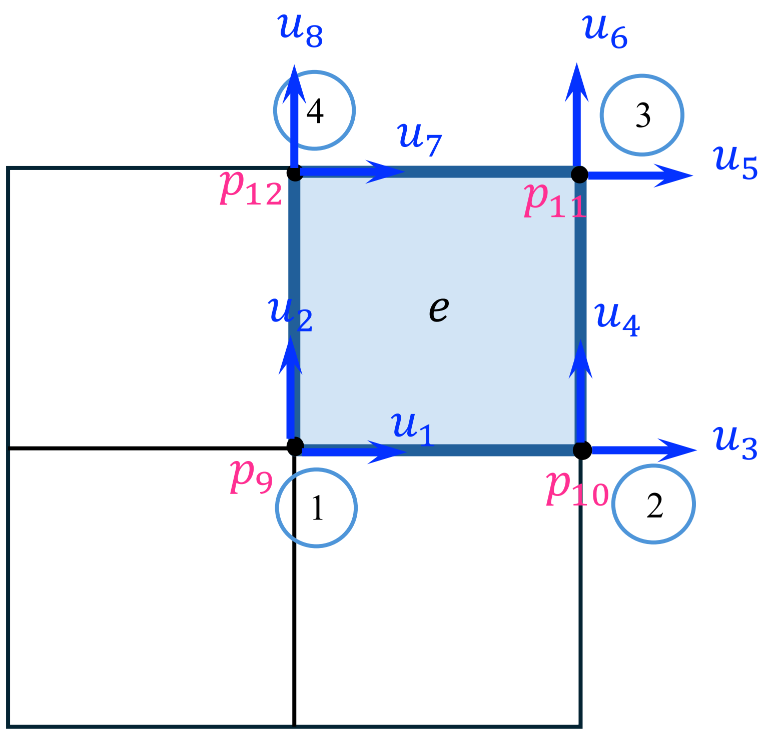
Fully implicit Euler method

FVM Staggered Discretization



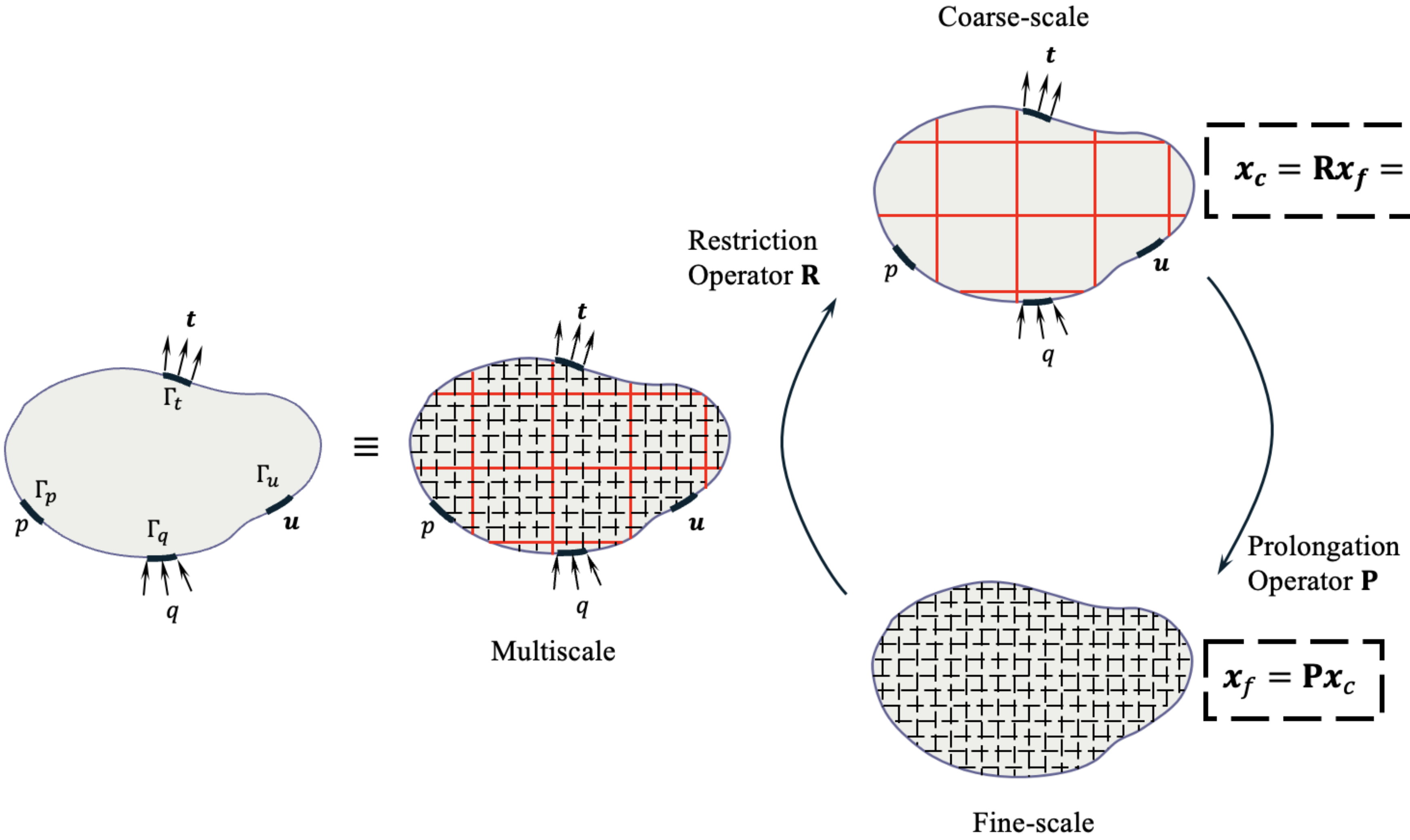
Cell-centered Pressure
Vertex-centered (Nodal) Displacement & Stress

FEM Galerkin Discretization



Nodal Displacement & Pressure

The Multiscale Framework



$$\boldsymbol{x}_c = \mathbf{R}\boldsymbol{x}_f = (\mathbf{RAP})^{-1}\mathbf{RF}$$

$$\boldsymbol{x}_f = \mathbf{P}\boldsymbol{x}_c$$

Basis Functions

$$\begin{cases} \nabla.\left(\mathbf{C} : \nabla^s \Phi^i_{\boldsymbol{u}_j}\right) = 0 & in \;\; \Omega^{\mathrm{P}}_j \\ \nabla_{||}.\left(\mathbf{C}_r : \nabla^s_{||} \Phi^i_{\boldsymbol{u}_j}\right) = 0 & in \;\; \partial\Omega^{\mathrm{P}}_j \\ \Phi^i_{\boldsymbol{u}_j}(x_k) = \delta_{ik} & \forall x_k \in \{1, \dots, N^H_{\boldsymbol{u}}\} \end{cases}$$

$$\begin{cases} -\nabla.\left(\boldsymbol{\Lambda} \,.\, \nabla \Phi^i_{\boldsymbol{p}_j}\right) = 0 & in \;\; \Omega^{\mathrm{D}}_j \\ -\nabla_{||}.\left(\boldsymbol{\Lambda} \,.\, \nabla_{||} \Phi^i_{\boldsymbol{p}_j}\right) = 0 & in \;\; \partial\Omega^{\mathrm{D}}_j \\ \Phi^i_{\boldsymbol{p}_j}(x_k) = \delta_{ik} & \forall x_k \in \{1, \dots, N^H_{\boldsymbol{p}}\} \end{cases}$$

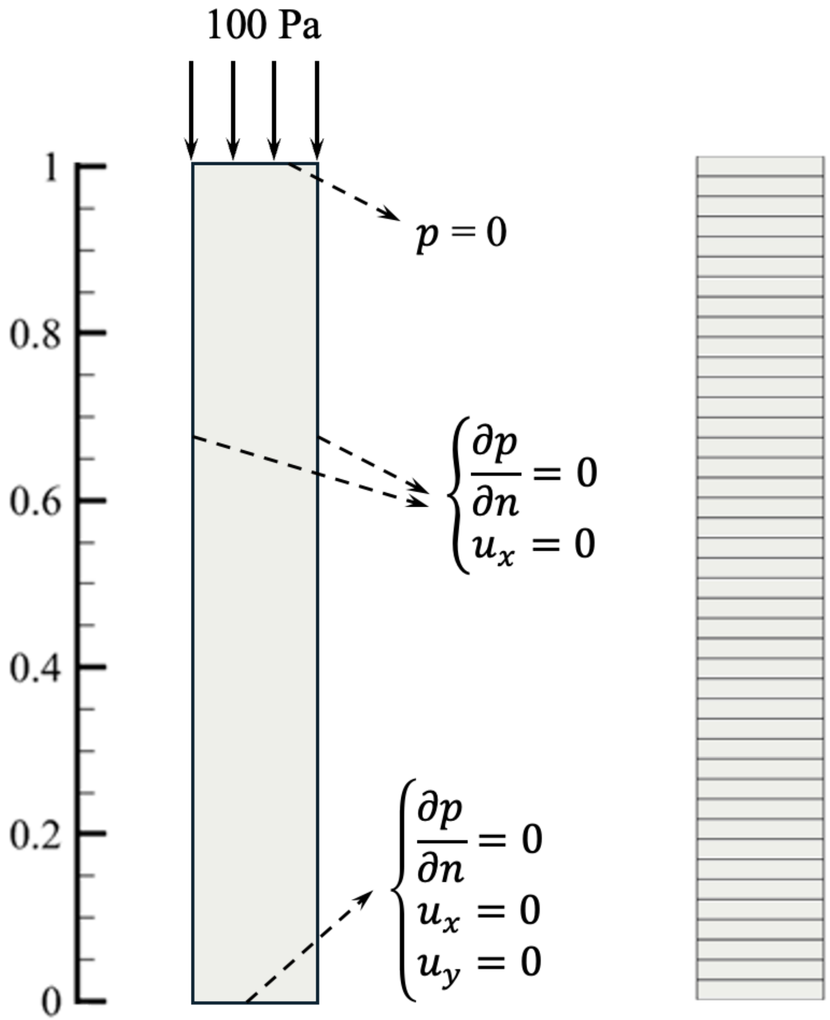
Coarsening Ratio ↓ Finer resolution & more accurate results

The Multiscale Framework

Coarse Scale		Fine Scale
Generate coarse grid for displacement (Primal) & pressure (Dual) independently,		Define fine-scale mesh,
Set boundary conditions and material properties,		Discretize using FV & FE on staggered or nodal grid,
For each time step n, **do**	→	Basis Function construction, $\nabla \cdot (\mathbf{C} : \nabla^s \mathbf{\Phi}_i^{u_j}) = 0$ in Ω_j^{P}; $-\nabla \cdot (\mathbf{\Lambda} \nabla \mathbf{\Phi}_i^{p_j}) = 0$ in Ω_j^{D}
Assemble Prolongation **P** & Restriction **R** from locally computed basis functions,	← $\mathbf{\Phi}_i^{u_j}$ & $\mathbf{\Phi}_i^{p_j}$ upscaling	
Solve coarse system, $\mathbf{x}_c = (\mathbf{R}\mathbf{K}_f\mathbf{P})^{-1}\mathbf{R}\mathbf{F}_f$		Assemble fine-scale fully implicit system, $\underbrace{\begin{bmatrix} \boldsymbol{K}_{uu} & -\boldsymbol{Q}_{up} \\ \frac{1}{\Delta t}\boldsymbol{Q}_{pu} & \boldsymbol{R}_{pp} + \frac{1}{\Delta t}\boldsymbol{S}_{pp} \end{bmatrix}}_{\mathbf{K_f}} \underbrace{\begin{Bmatrix} \boldsymbol{u}^{(n+1)} \\ \boldsymbol{p}^{(n+1)} \end{Bmatrix}}_{\mathbf{x_f}} = \underbrace{\begin{Bmatrix} \boldsymbol{F}_u \\ \boldsymbol{F}_p + \frac{1}{\Delta t}\left(\boldsymbol{S}_{pp}\boldsymbol{p}^{(n)} + \boldsymbol{Q}_{pu}\boldsymbol{u}^{(n)}\right) \end{Bmatrix}}_{\mathbf{F_f}}$
Reconstruct fine-scale $\mathbf{x}_f \approx \mathbf{P}\mathbf{x}_c$	→ downscaling	Approximate fine-scale solution, $\mathbf{x}_f \approx \mathbf{P}\mathbf{x}_c = \begin{Bmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{Bmatrix}$
End for		

Numerical Simulation Results

1D Terzaghi Benchmark

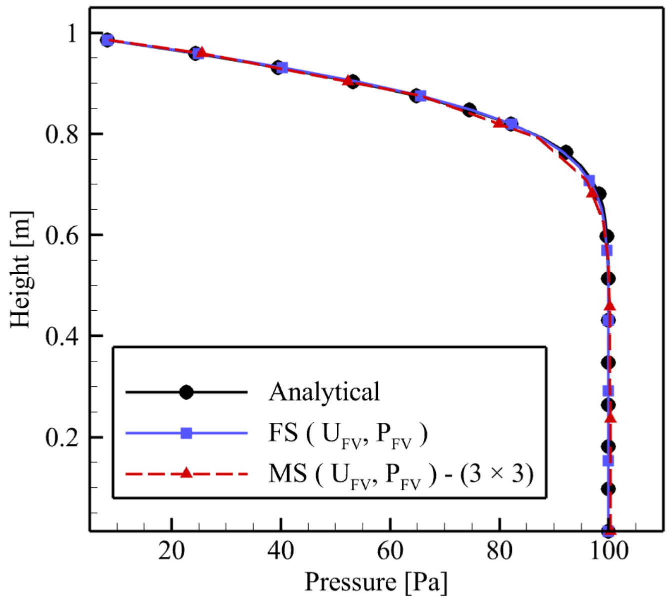


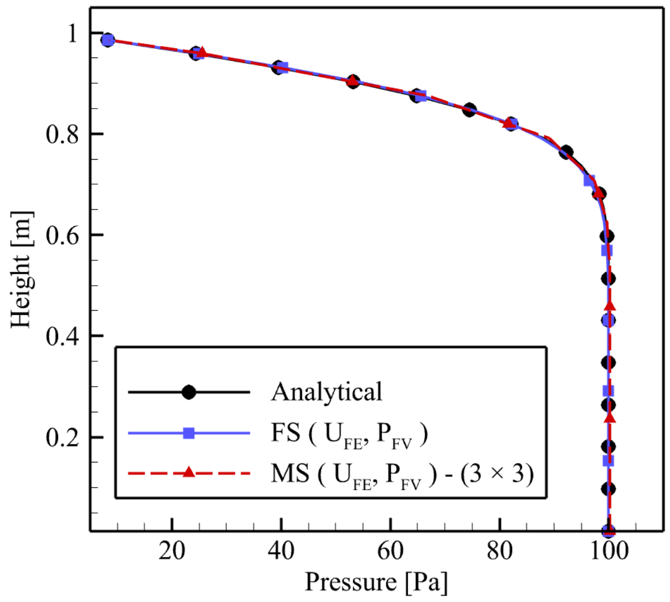
1×42 fine-scale mesh

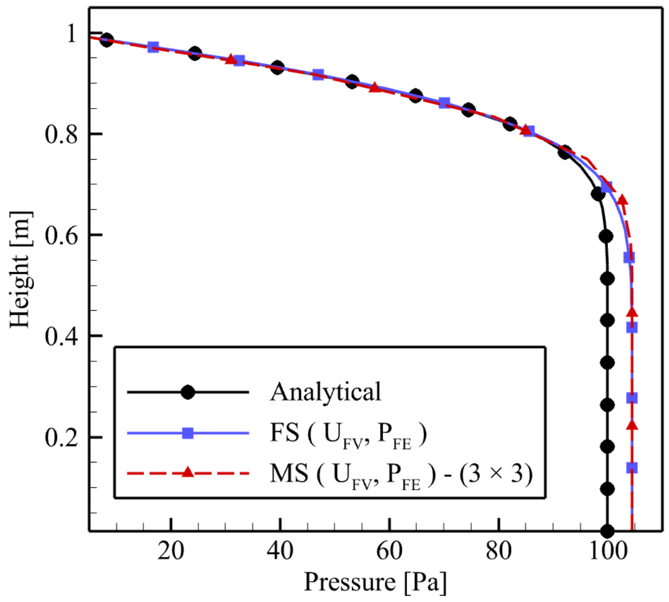
Property	Value
Young modulus (E)	10^4 Pa
Poisson's ratio (ν)	0.2
Biot's Modulus (Q)	10^{100} Pa
Rock permeability (K)	10^{-7} m^2
Fluid viscosity (μ_f)	10^{-3} Pa · s
Biot's coefficient (α)	1

Numerical Simulation Results

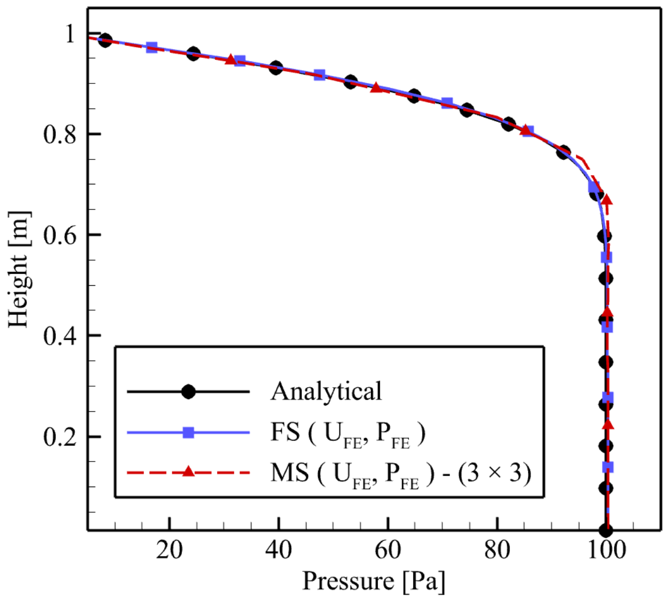
Pressure Distribition along Terzaghi Column







MS: Coarsening Ratio of 3



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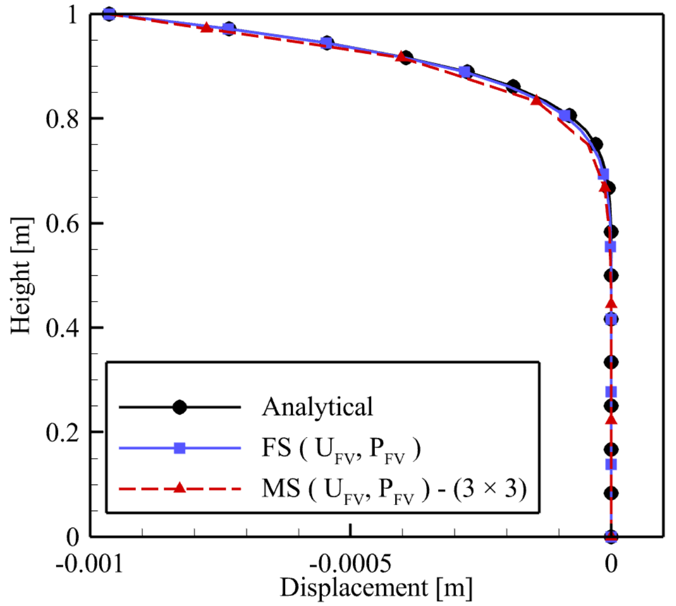
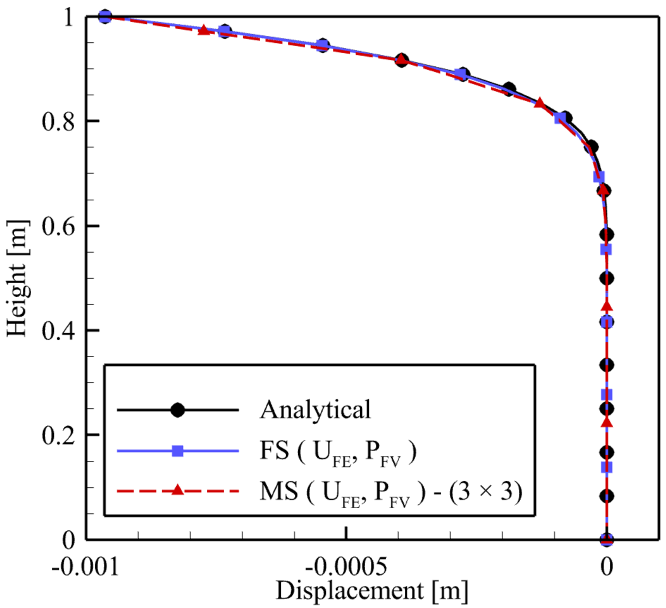
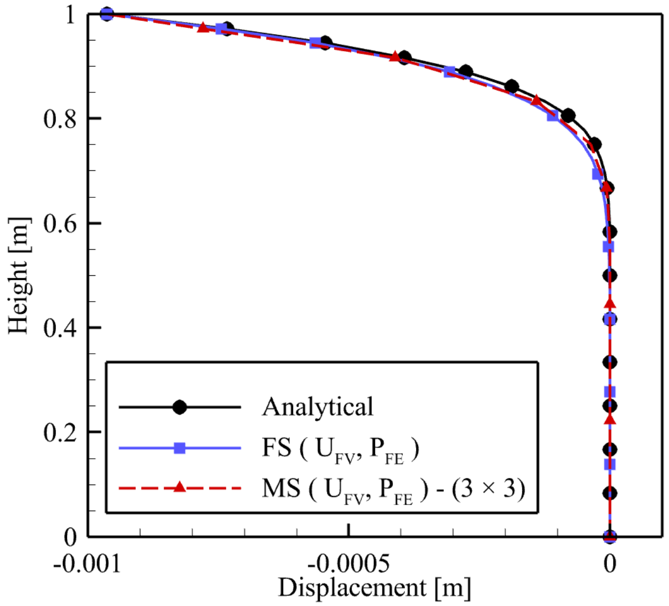
Numerical Simulation Results

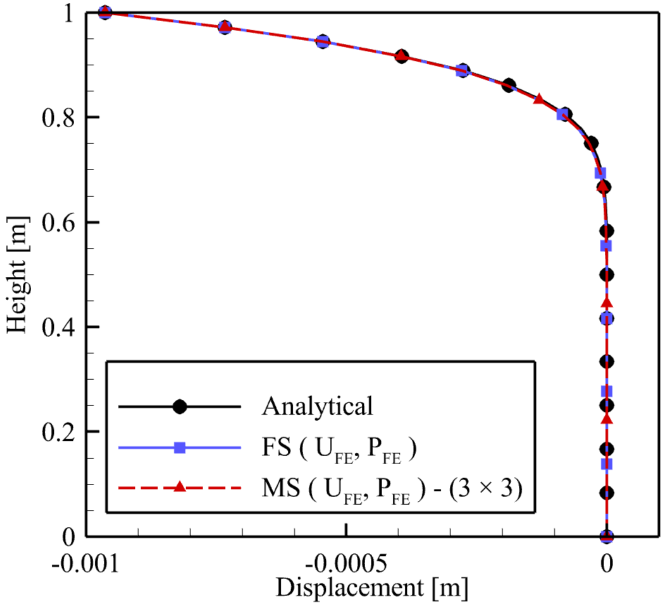
Displacement Distribition along Terzaghi Column







MS: Coarsening Ratio of 3



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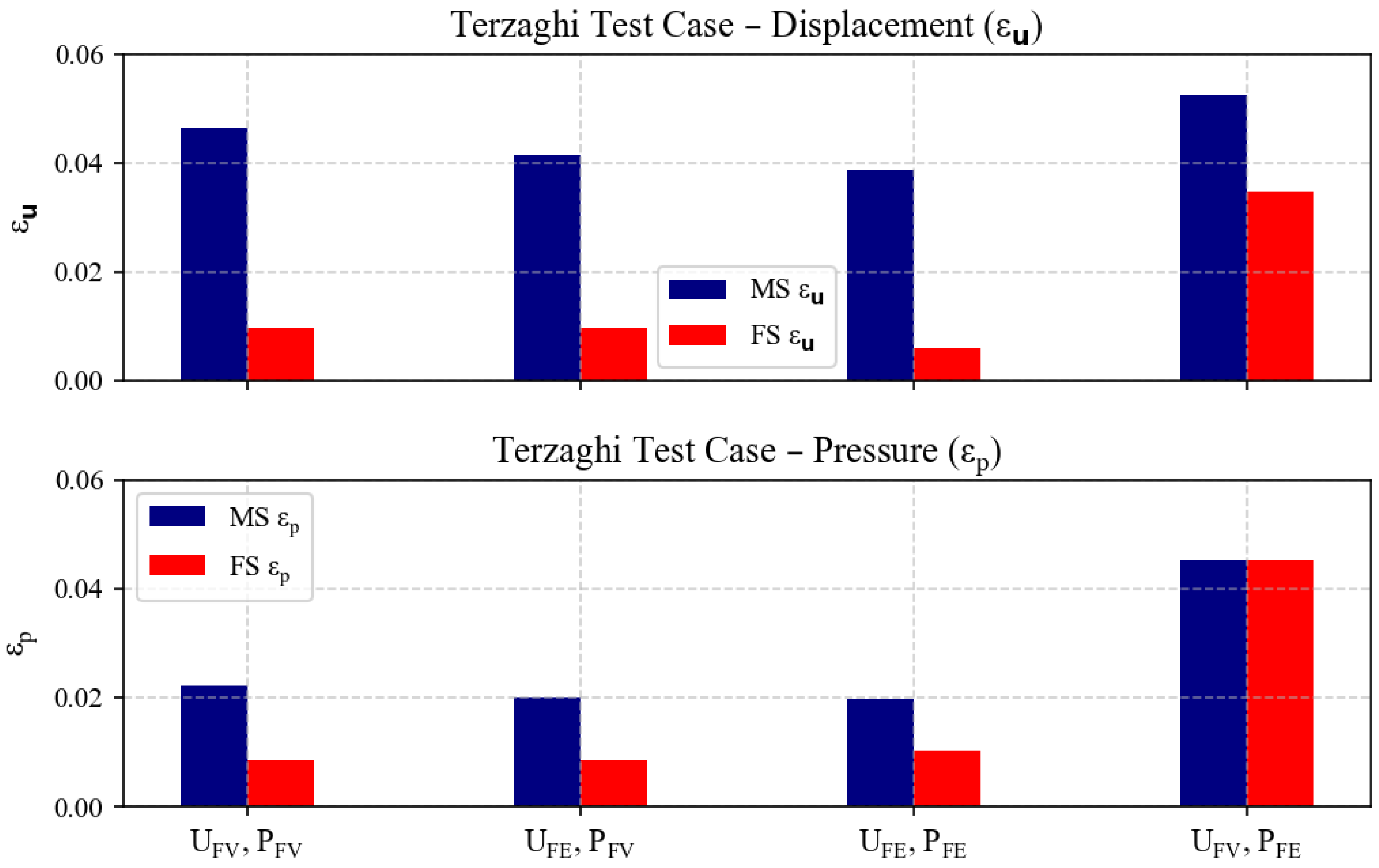
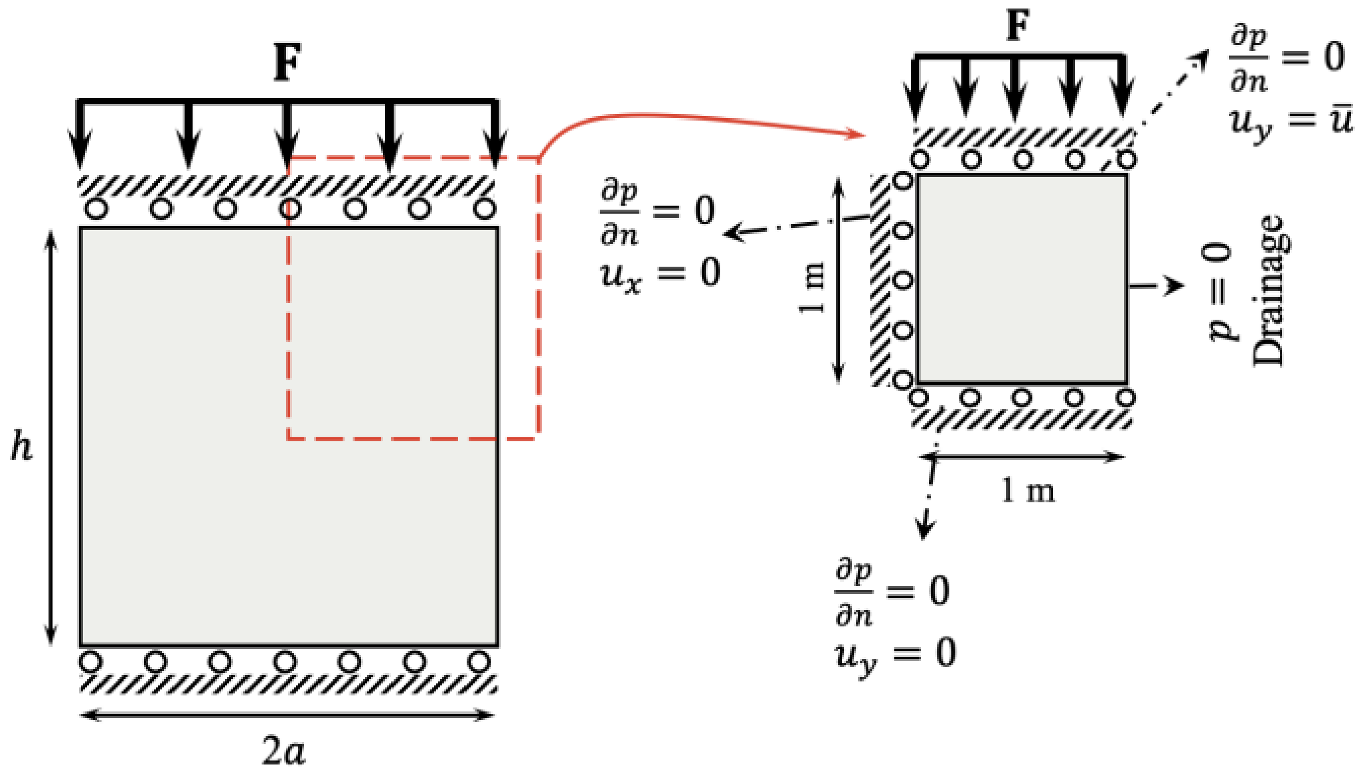
Numerical Simulation Results



Comparison of the multiscale (MS) and fine-scale (FS) solutions against the analytical solution for displacement and pressure errors under different variable formulations.

Numerical Simulation Results

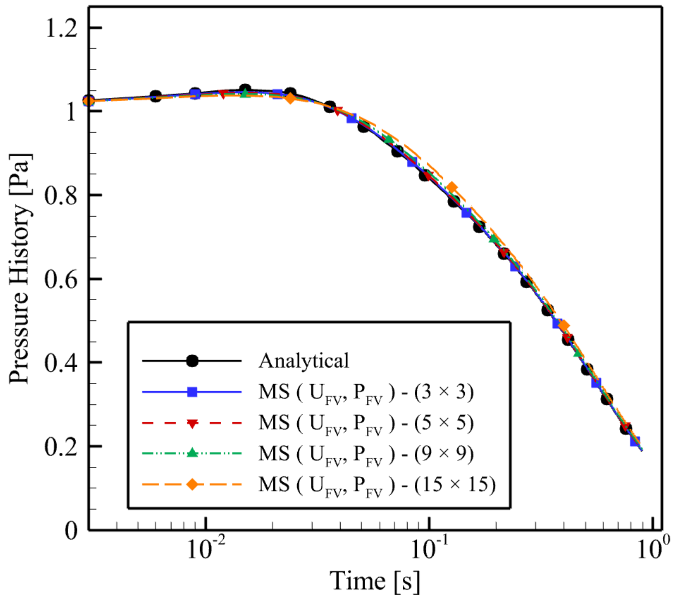
2D Mandel Benchmark

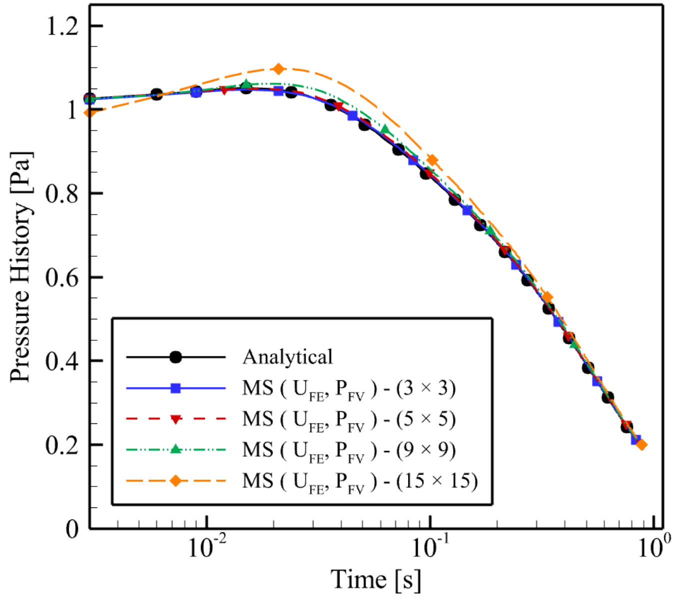


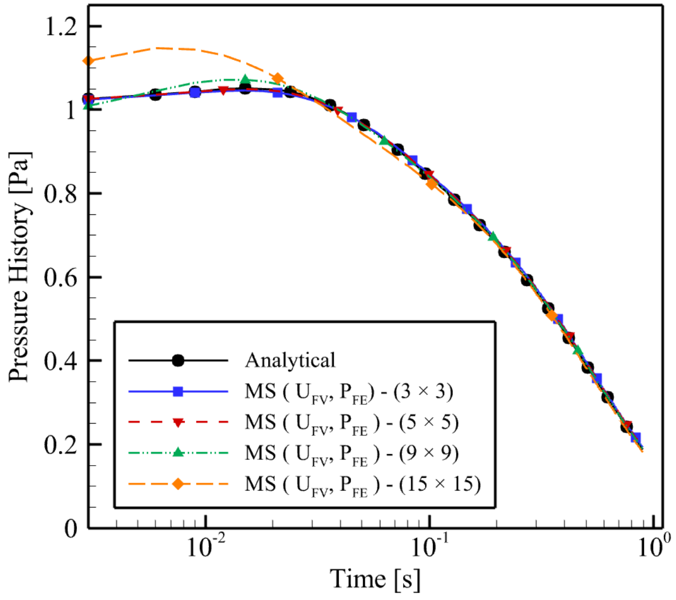
Property	Value
Young's modulus (E)	10^4 Pa
Poisson's ratio (ν)	0.2
Biot's coefficient (b)	1
Biot's modulus (M_b)	10^{10}
Permeability (k)	10^{-7} m^2
Fluid viscosity (μ_f)	10^{-3} Pa · s

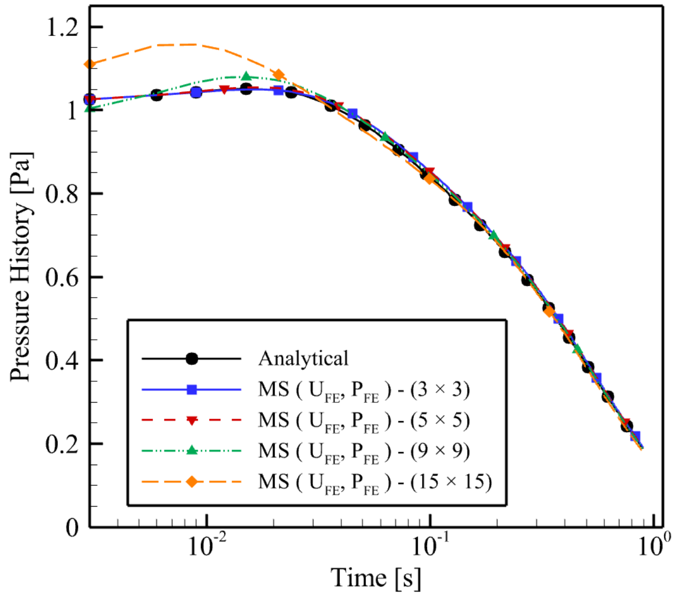
Numerical Simulation Results

Temporal Variation of Pore Pressure @ domain central point









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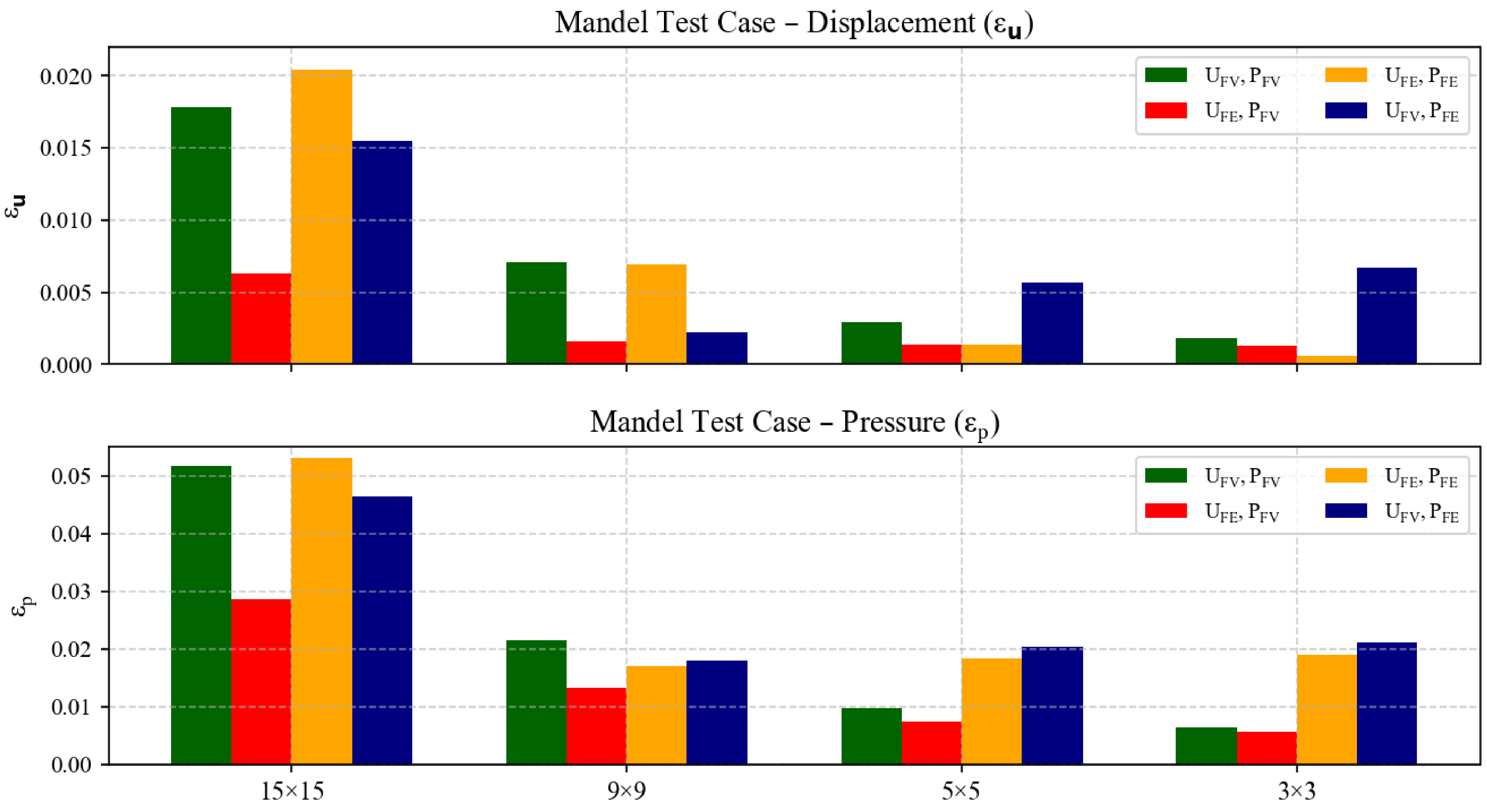
Numerical Simulation Results



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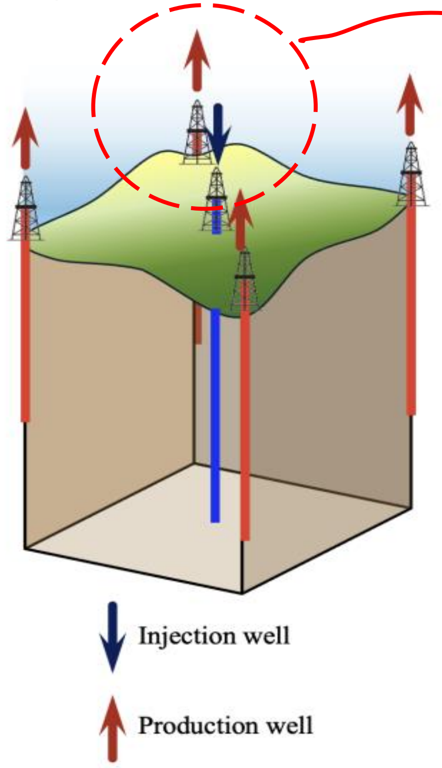
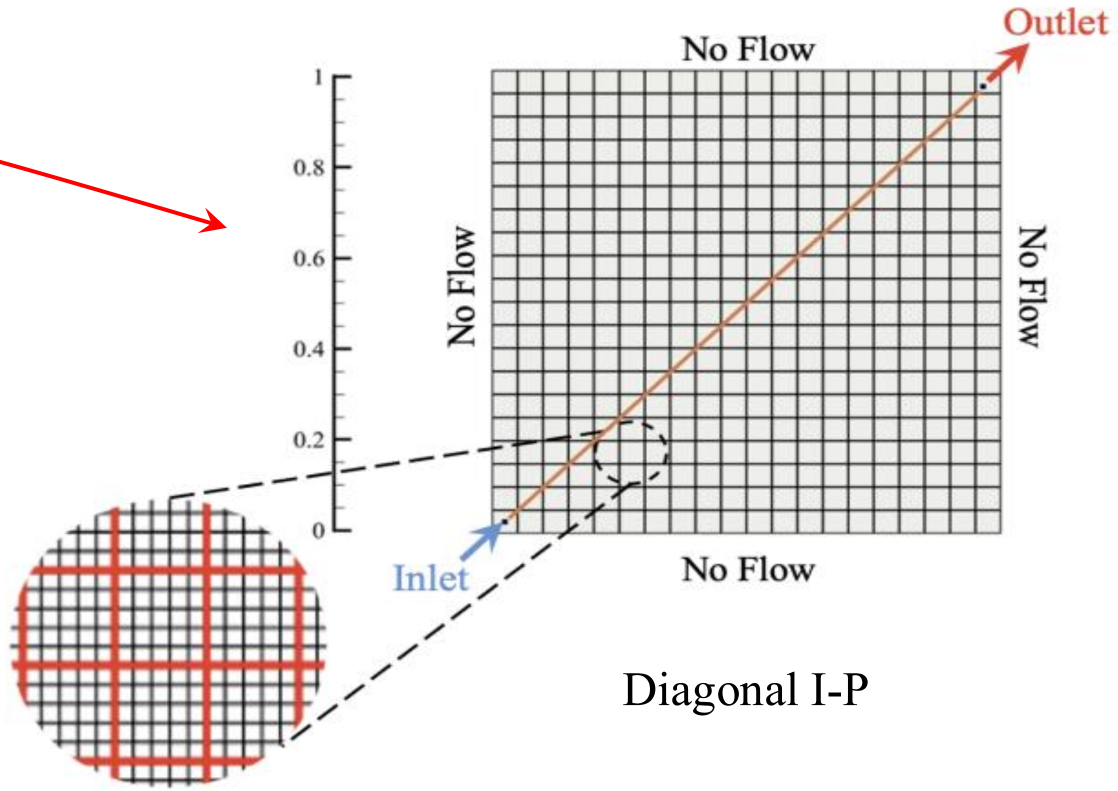
Numerical Simulation Results

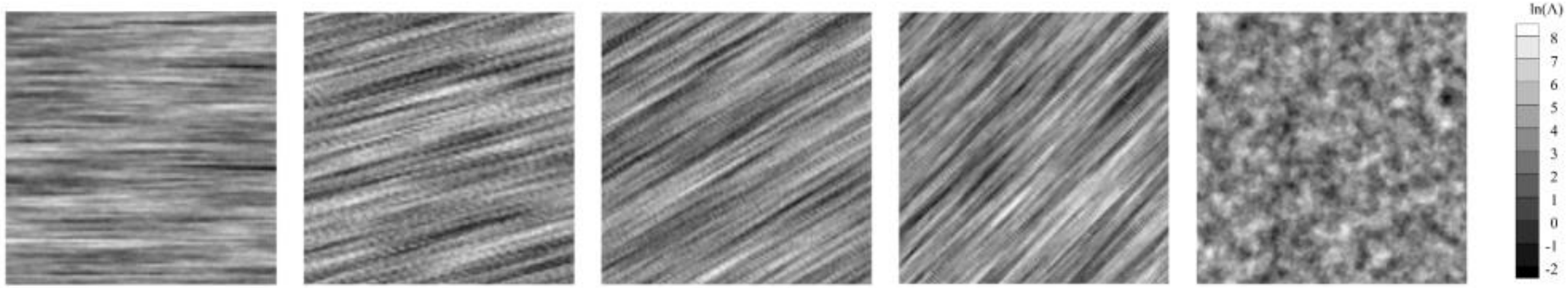




Diagonal I-P

θ = 0 ∘ 15∘ 30∘ 45∘ non-directional

$$-\nabla \cdot (\lambda \cdot \nabla p) = q$$



Natural logarithm of the heterogeneous mobility field ln(Λ) of 20 realizations

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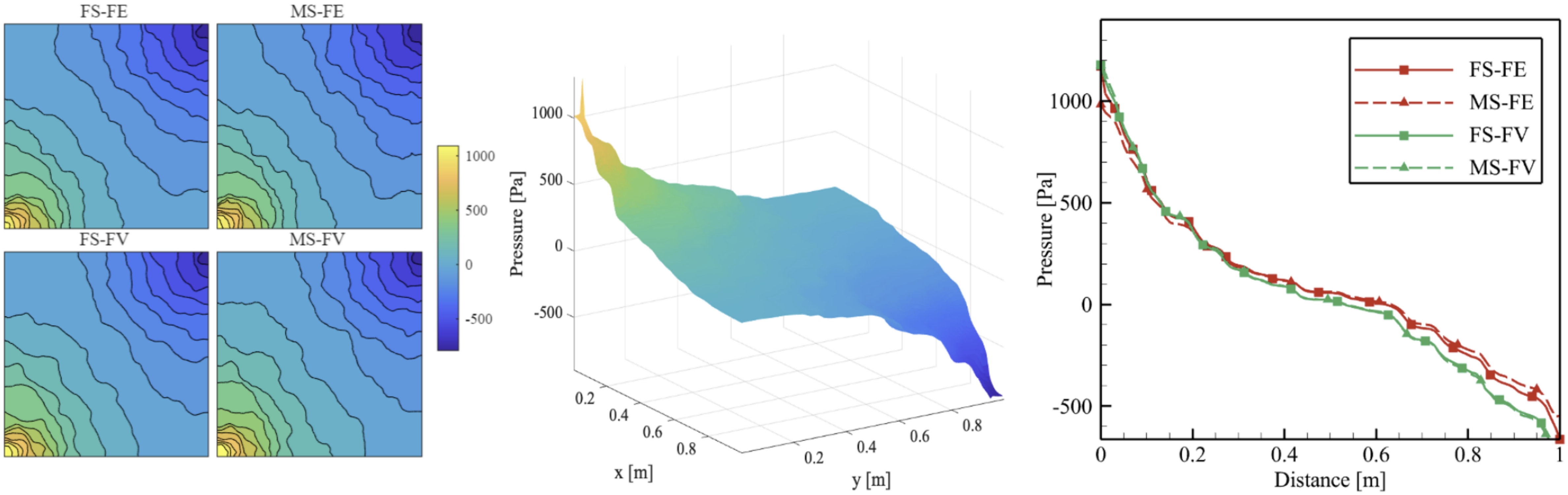
Numerical Simulation Results

MS: Coarsening ratio of 5



(a) (b) (c)

Pressure fields for a test case with irregularly distributed heterogeneity; (a) contour maps obtained with the FS-FE, MSFE, FS-FV, and MSFV schemes, (b) corresponding 3D pressure surface, and (c) pressure profile along the diagonal line connecting the injection and production wells. The MS simulations use a coarsening ratio of 5 × 5 . Pressure along the well-to-well line is evaluated by linear interpolation, with distance normalized from 0 to 1.

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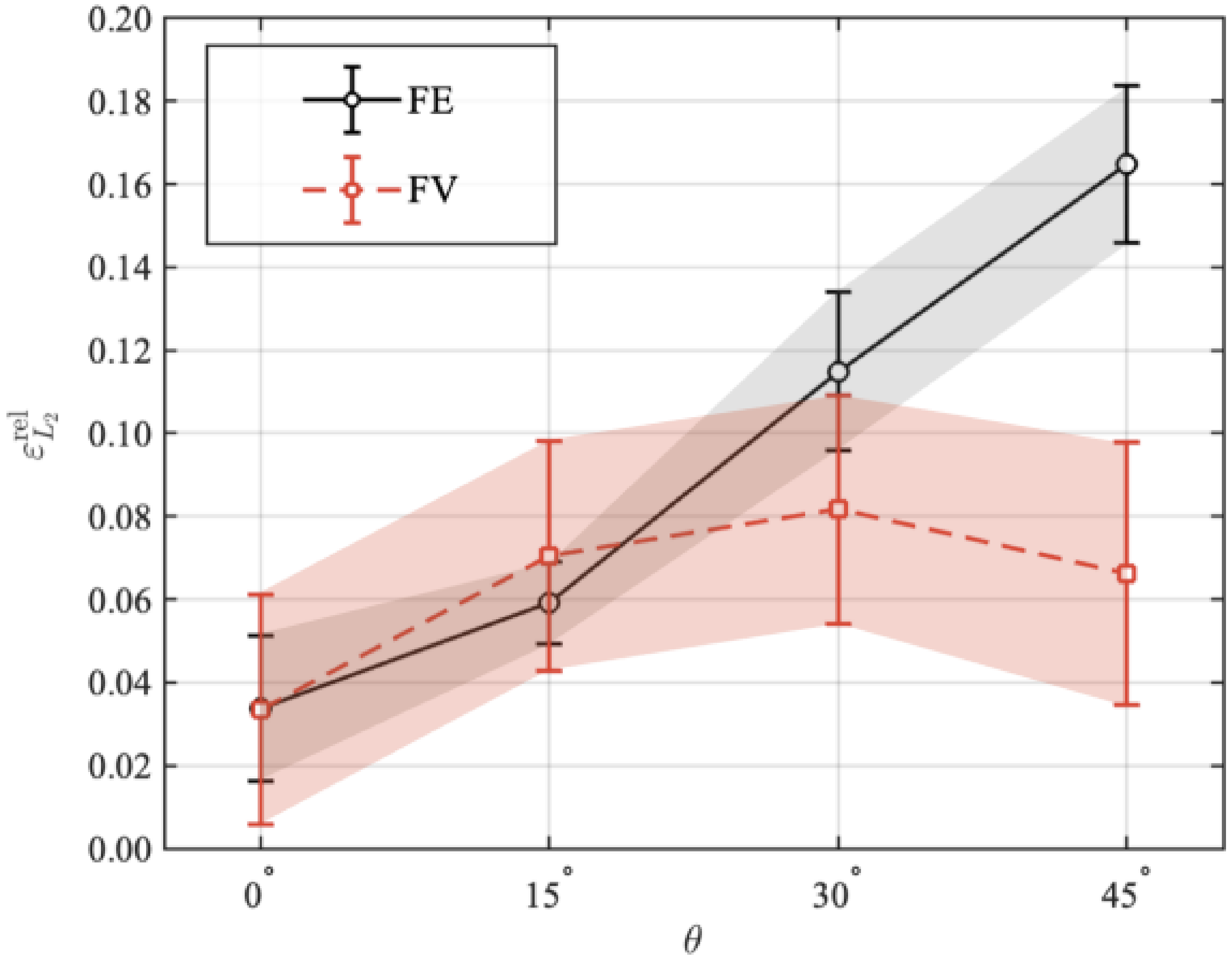
Numerical Simulation Results



Conclusions

- Both the MSFV and MSFE formulations accurately reproduce fine-scale reference solutions across all benchmark tests, demonstrating the robustness of the unified multiscale framework.

- The FV formulation on a staggered grid guarantees local mass and stress conservation and remains stable even under high coarsening ratios. The FE formulation provides smooth displacement fields thanks to its continuous interpolation.

- The hybrid combination, using finite elements for displacement and finite volumes for pressure, offers the best overall balance: lowest errors, good conservation, and competitive computational cost.

- For anisotropic heterogeneous flow problems, the FV-based multiscale is more robust to varying permeability orientations.

Q & A

Thanks for your attention!

m.mehrazar@tudelft.nl