Numerical Modeling of Infiltration using a Bounded Auxiliary Variable

Abderrahmane Benfanich Yves Bourgault Abdelaziz Beljadid

University of Ottawa, Department of Mathematics and Statistics

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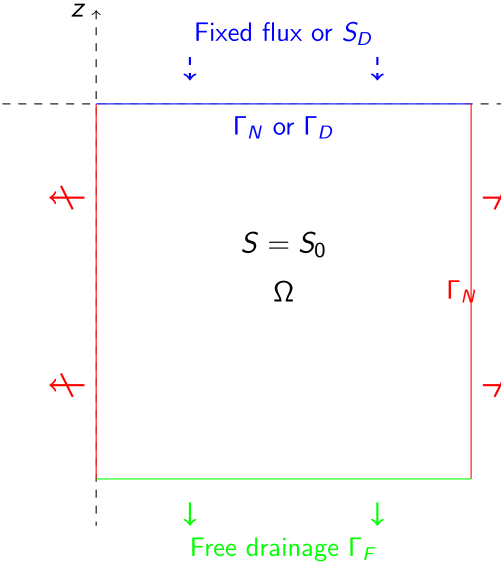
Outline

Motivation: Infiltration in Porous Media

- Water infiltration through unsaturated soils is central to **hydrology**, environmental engineering, and material science.
- Governed by the **Richards equation** (Richards 1931) — a nonlinear elliptic–parabolic PDE.
- Challenge: domains may contain **both fully dry** ($S = 0$) **and fully saturated** ($S = 1$) regions simultaneously.
- Classical formulations *cannot* handle both limits without **variable switching** or **regularization**.



The Richards Equation

Governing Equation

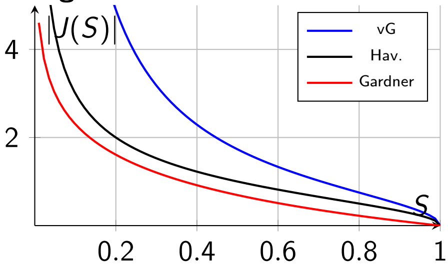
$$\frac{\partial \theta}{\partial t} + \nabla \cdot \boldsymbol{q} = 0, \qquad \boldsymbol{q} = -K_s K_r(S)\, \nabla(\Psi(S) + z),$$

where $\theta = \phi\, S$ (water content), S = effective saturation, K_s = saturated conductivity, K_r = relative permeability, Ψ = capillary pressure head.

Three classical formulations:

- Ψ-form: valid at $S = 1$, *unstable as* $S \to 0$
- S-form: valid at $S = 0$, *degenerate at* $S = 1$ (needs regularization)
- Mixed (S, Ψ)-form: better mass conservation but inherits the limitations above

Leverett function $J(S) = \Psi(S)/h_{\text{cap}}$ diverges:



Key Idea: Introducing the Auxiliary Variable u

Root cause of difficulty: the derivative $J'(S)$ is singular at $S = 0$ (for Gardner, Brooks–Corey models) *and* at $S = 1$ (for van Genuchten, Haverkamp models).

For all standard models, $J'(S) = C\, S^{-a}(1 - S^c)^{-b}$. Define

$$\boxed{u = \int_0^S (1 - s^c)^{-b}\, ds = \frac{1}{c}\, \mathcal{B}\big(S^c, \tfrac{1}{c}, 1 - b\big)}$$

- u is **monotone** for $S \in [0, 1]$ and **bounded**: $0 \le u \le u_{\max} < \infty$.
- Rewriting the Richards equation in terms of u *removes all unbounded terms* from the weak formulation.
- **No variable switching. No regularization parameter.**

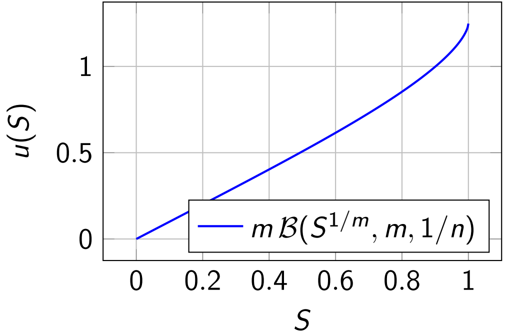
The u-Formulation

Richards equation in u (homogeneous medium)

$$\phi \frac{\partial S(u)}{\partial t} - \nabla \cdot \Big(K_s\, K_r(S(u)) \Big(h_{\text{cap}}\, C\, S(u)^{-a}\, \nabla u + \boldsymbol{e}_z \Big) \Big) = 0$$

All terms stay bounded because
$\lim_{S \to 0} K_r(S)\, S^{-a} < \infty$ for van Genuchten–Mualem model ($n > 1$) and other standard models.

Model	$u(S)$
Gardner	S
Brooks–Corey	S
Haverkamp	$\beta[1 - (1 - S)^{1/\beta}]$
van Genuchten	$m\, \mathcal{B}(S^{1/m}, m, 1/n)$



Graph of u vs. S (van Genuchten model)

Finite Element Discretization

Space: linear Galerkin FEM on a conforming mesh.
Time: semi-implicit Euler; two variants:

Nonlinear semi-implicit scheme (Newton iterations)

Find $u_h^{n+1,m+1} \in V_h$ such that $\forall v_h \in V_h$:

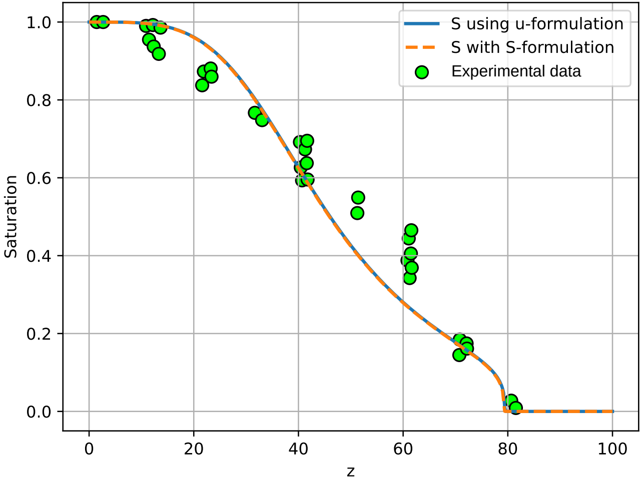
$$\int_\Omega \phi \frac{\partial S}{\partial u}(u_h^{n+1,m}) \frac{u_h^{n+1,m+1} - u_h^{n+1,m}}{\Delta t} v_h \, dx + \int_\Omega \phi \frac{S(u_h^{n+1,m}) - S(u_h^n)}{\Delta t} v_h \, dx$$

$$+ \int_\Omega K_s K_r(S(u_h^n)) \left(h_{\text{cap}} S(u_h^n)^{-a} \nabla u_h^{n+1,m+1} + \boldsymbol{e}_z \right) \cdot \nabla v_h \, dx + (\text{BC terms}) = 0.$$

- **Mass lumping** on the temporal term $\Rightarrow$ monotone saturation, physical bounds preserved.
- Nonlinear solver converges in **1–5 Newton iterations** per time step in all tested configurations.
- A *linear* semi-implicit variant (one Newton step) has similar speed as the regularized S-formulation without the need for choosing a regularization parameter.

Test 1 — Fibrous Sheet: Fully Dry Initial Medium

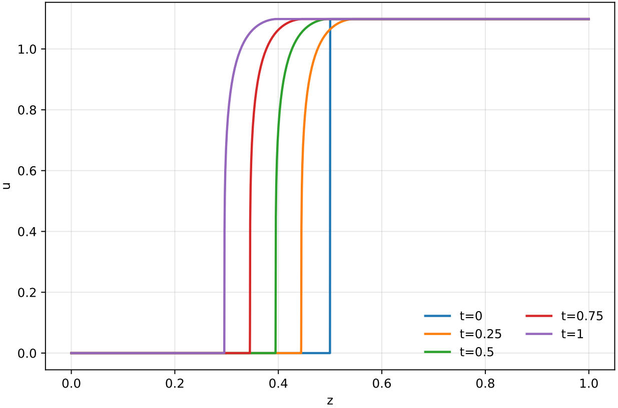
- 1D mineral-oil flow in fibrous sheet (Jaganathan et al. 2009).
- Initial condition: $S(z, 0) = 0$ (completely dry).
- Boundary: $S = 1$ at bottom, $S = 0$ at top.
- **Classical S-form works here** (no fully saturated region), so agreement validates u-form.
- Both formulations and experimental data match excellently.



u- and S-formulations vs. experimental data (Jaganathan 2009)

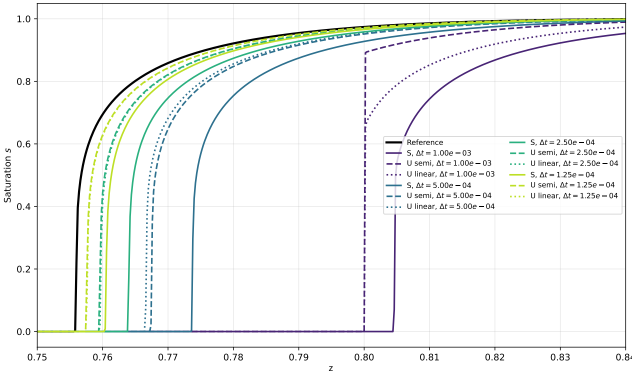
Test 2 — Mixed Saturated/Dry Regions (1D)

- Initial condition: $S_0 = 0$ on $[0, 0.5]$, $S_0 = 1$ on $(0.5, 1]$.
- Neither S- nor Ψ-formulation applies without switching/regularization.
- **u-formulation handles it directly.**
- Nonlinear semi-implicit scheme; 5000 mesh points; $\Delta t = 10^{-5}$ hours.
- Monotone, positive solution throughout.



u-solution with $h = 2 \times 10^{-4}$, $\Delta t = 10^{-5}$ *hours*

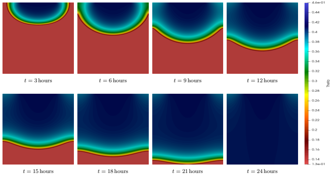
Test 2 — Comparison with Regularized S-Formulation



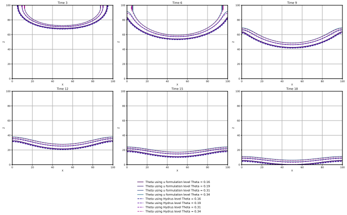
Sharp-front resolution for various Δt

- S-form (solid): regularization parameter δ causes **artificial smoothing** at the front.
- u semi-implicit (dashed) and u-linear (dotted): **sharp interface preserved**.
- u-formulation avoids the problem of choosing δ: too large $\Rightarrow$ lag; too small $\Rightarrow$ ill-conditioning.
- u-linear matches S-form in CPU time with **no tuning required**.

Test 3 — 2D Fully Saturated Region



Water content θ at various times



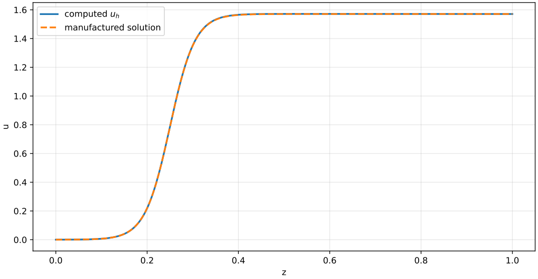
Contours: u-formulation vs. Hydrus

- 100×100 cm^2; Dirichlet inflow strip on top
- $\theta_0 = 0.13$, $\theta_D = 0.41$ (fully saturated)
- 95,142 triangles; $\Delta t = 0.001$ h
- Linear semi-implicit u-form is sufficient
- Results agree closely with **Hydrus** software

Convergence and Accuracy

Manufactured solution (van Genuchten – Mualem model):

- Exact solution prescribed; source term added.
- **Space order** $p \approx 2$ on finest meshes.
- **Time order** $q \approx 1$ (semi-implicit Euler).



Numerical vs. exact solution at $t = 1$ day

Space convergence ($\Delta t = 5 \times 10^{-5}$):

h	L^2 error	p
1.43×10^{-2}	6.04×10^{-3}	—
7.14×10^{-3}	8.99×10^{-4}	2.75
3.57×10^{-3}	1.39×10^{-4}	2.69
1.79×10^{-3}	3.48×10^{-5}	2.00

Time convergence ($h = 10^{-3}$):

Δt	L^2 error	q
10^{-2}	8.23×10^{-3}	—
5×10^{-3}	3.90×10^{-3}	1.08
2.5×10^{-3}	1.92×10^{-3}	1.02
1.25×10^{-3}	9.60×10^{-4}	1.00

Conclusion

Summary

- We introduced a **bounded auxiliary variable** u that removes all unbounded terms from the Richards equation weak formulation.
- A **semi-implicit FEM** with Newton iterations solves the resulting nonlinear system — no regularization, no variable switching.
- **Mass lumping** preserves physical bounds and monotonicity.
- Numerical solutions remain **positive even in completely dry zones** ($S = 0$).
- The method is validated against experimental data (fibrous sheets), Hydrus (2D saturated flow), and manufactured solutions.

Ongoing and future work

- Rigorous mathematical analysis (existence, bound preservation, convergence) established in Benfanich et al. (2026). arXiv:2602.19037
- Extensions to coupled solute and heat transport (multiphysics applications).

Thank you!

abenf099@uottawa.ca

Department of Mathematics and Statistics, University of Ottawa

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