Quantification of Heterogeneity of Spatially Averaged Generalized sub-Gaussian Random Fields

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Outline

- ➢ Generalized sub-Gaussian (GSG) random fields
- ➢ Spatially averaging random fields
- ➢ Spatially averaged GSG variance model
- ➢ Model Verification: MC Results
- ➢ Application to data

Generalized sub-Gaussian Random Fields

$Y = U\,G$

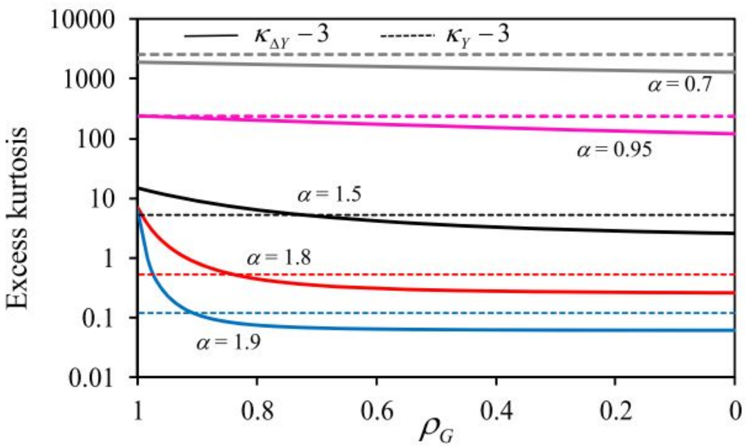
U - Log-normal subordinator ($\mu_U = 0, \sigma_U = (2\text{-}\alpha)$)

G - Gaussian Random Field ($\mu_G = 0,\ \sigma_G, \rho_G$)

$\Delta Y = Y(x) - Y(x+h)$

x - Spatial Coordinate

h - Separation distance



"No statistical model known to us captures these behaviors of Y and ΔY in a unified and consistent manner."

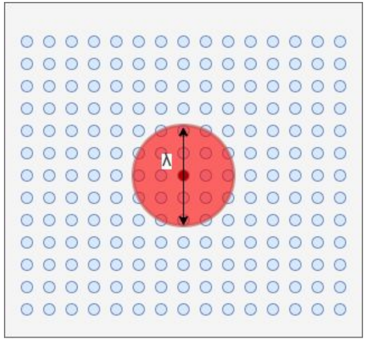
1. Riva, M., Neuman, S. P., & Guadagnini, A. (2015). New scaling model for variables and increments with heavy-tailed distributions. *Water Resources Research*, *51*(6), 4623-4634.

Spatially Averaging Random Fields

- Weighted Spatial Average Definition

$$< Y(x) >_\lambda = \int_{\mathbb{R}^d} w_\lambda(x + x')Y(x + x')dx \qquad w_\lambda(x) = \frac{1}{\lambda^d} w_1\left(\frac{x}{\lambda}\right)$$

- Uniform weights: Moving Frame Window
    - Circle of radius, λ, centered on each data point. All data points within the circle are arithmetically averaged.



Spatially Averaged GSG Variance Model

➤ General Model:

- Inner integral is the convolution of 2 weight functions - simple to evaluate for specific choices of weights

$$\mathbb{E}\left[< Y(x) >^2_\lambda\right] = \mathbb{E}[U]^2 \int_{\mathbb{R}^d} c_G(|u|) \int_{\mathbb{R}^d} w_{\lambda 1}(y' + u) w_{\lambda 2}(y')\, dy du$$

➤ Model Specialisation

- 2D uniform weights, exponential covariance form.

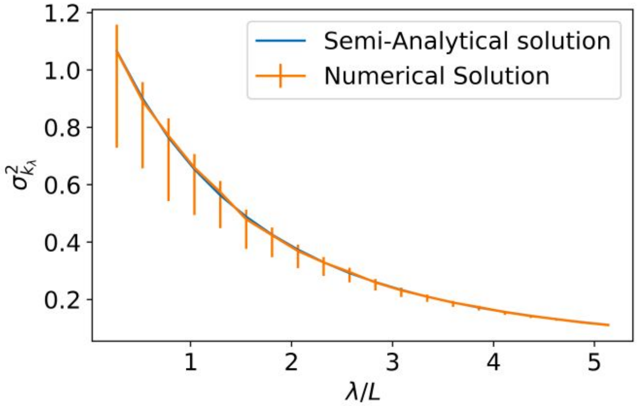
$$\mathbb{E}[< Y(x) >^2_\lambda] = \frac{4\exp((2-\alpha)^2)\sigma_G^2}{\lambda^2 \pi} \int_0^{2\lambda} \exp(-\tfrac{r}{l}) \left(\arccos\left(\tfrac{r}{2\lambda}\right) - \tfrac{r}{2\lambda}\sqrt{1 - \tfrac{r^2}{4\lambda^2}} \right) r\, dr$$

Model Verification: Monte Carlo Results

➤ MC Results Vs Analytical Solution: Highly leptokurtic test case

- GSG fields generated using Scipy and Numpy in Python.
- Spatially averaged using uniform weights and a moving frame window average.
- Largest error <2%.

Parameter	Value
α	1.5
σ_G	1
Integral Length Scale (L)	1
MC simulations	5000
Domain Size	10x10
Grid Size	50x50
λ Range	0.21 - 4

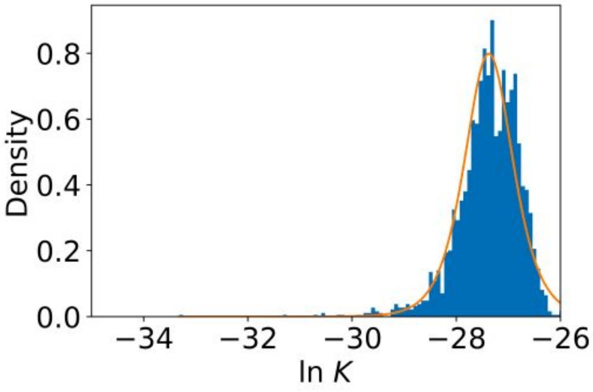


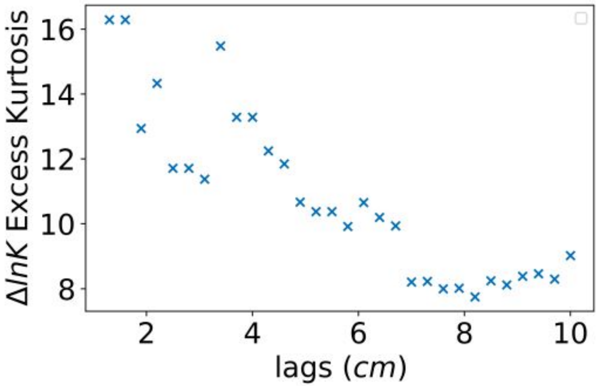
Application to Data

➤ Maximum Likelihood

- Massillon Sandstone block (0.94x0.96x1.01m) from Briar Hill Stone Company, Ohio USA.
- Uniform grid sampling on 6 sides of the block. 75000 Measurements in total.
- Multi Measurement Scale dataset - varying tip seal radius of permeameter (0.15, 0.31, 0.63, 1.27, 2.54 cm).
- Parameters for analytical solution obtained via Maximum Likelihood.

Parameter	ML estimate (95% Confidence interval)
α	1.681 ($\pm$0.040)
σ_G	0.525 ($\pm$0.019)

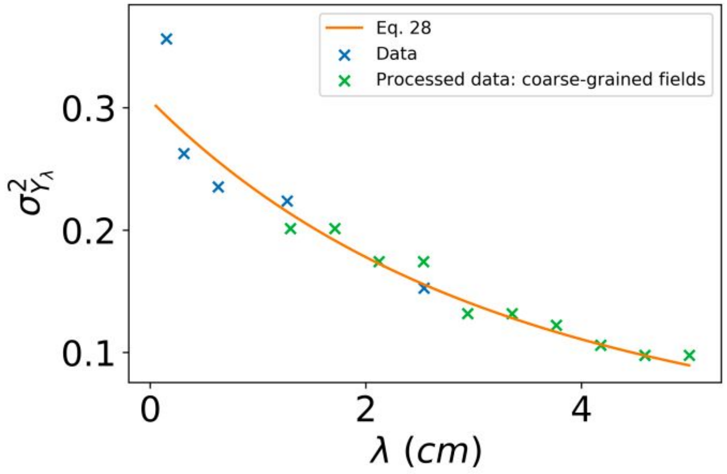




Application to Data

➢ Comparison of Hard Data with Variance Model

- 2D Analytical solution (orange) aligns well with multi-scale data (blue).
- Data manually spatially averaged with uniform weights to obtain a larger range of λ
- Excluding anomalous first data point (λ=0.15cm); max error <5%.



Conclusions

- ➢ Measurement scales of datasets should always be considered as it affects single- and multi- point statistics.

- ➢ Data collected at larger measurement scales is less heterogeneous as it is an average over the entire volume - high frequency fluctuations are smoothed.

- ➢ Our model allows a practitioner to determine the level of heterogeneity in their dataset in the context of the measurement length scale *and* allows estimations of heterogeneity at alternative length scales.

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