

# Positive DDFV scheme for degenerate parabolic equations arising from infiltration problem

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# Motivation

Nonlinear degenerate parabolic equations are the main core to study some complex problems arising, for instance, from petroleum engineering, hydrology and biology.

$$\partial_t u(t, x) - \nabla \cdot \left( a(u) \boldsymbol{\Lambda}(x) \nabla u(t, x) \right) = f \text{ in } Q_T := \Omega \times (0, T) \quad (1.1)$$

In the context of porous media flows, the diffusion function  $a$  is usually called the mobility while the tensor  $\boldsymbol{\Lambda}$  stands for the permeability.

## Porous medium equation

$$a(u) = u^m, m > 0. \quad (u \text{ is the density of a single phase})$$

## Richards' Equation

$$a(u) = \frac{1}{\mu_w} k_w(u) p'_c(u) \text{ is the mobility-capillary function, } u \text{ is the saturation of wetting phase}$$

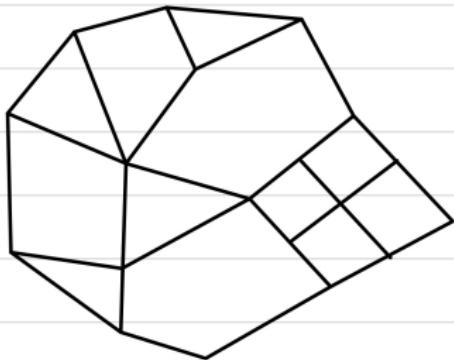
## Incompressible two phase flow

$$\phi \partial_t s - \operatorname{div}(\mathbf{K} a(s) \nabla s) + \operatorname{div}(\nu_w(s) \mathbf{V}_T) = 0$$

$$\text{with } a(0) = a(1) = 0$$

# Objective

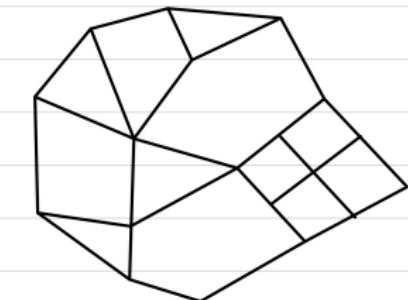
Construction of a positive scheme on general mesh combining triangles and any polygons structure



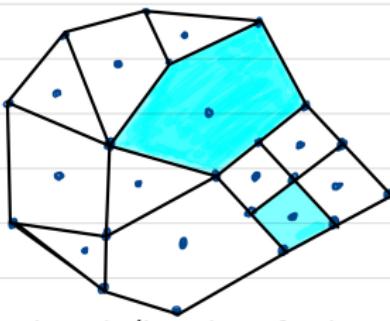
The tensor  $\Lambda$  is anisotropic.

# DDFV on general meshes

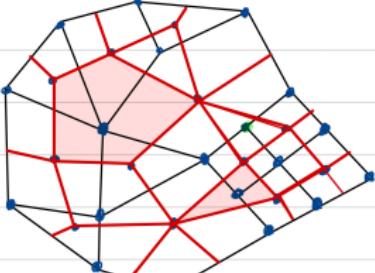
A DDFV mesh is made of  $\mathcal{T} = (\overline{\mathfrak{M}}, \overline{\mathfrak{M}}^*)$  and  $\mathfrak{D}$ . **The unknowns are located at the centers and at the vertices.** For every  $u_{\mathcal{T}} \in \mathbb{R}^{\#\mathcal{T}}$ , one has  $u_{\mathcal{T}} = ((u_K)_{K \in \overline{\mathfrak{M}}}, (u_{K^*})_{K^* \in \overline{\mathfrak{M}}^*})$ .



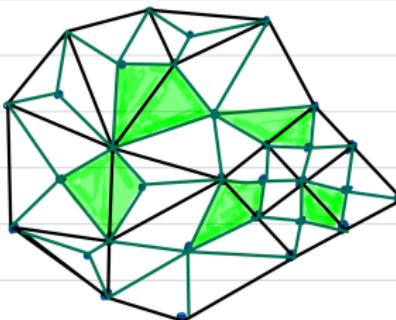
Initial mesh



Primal mesh (location of unknowns)

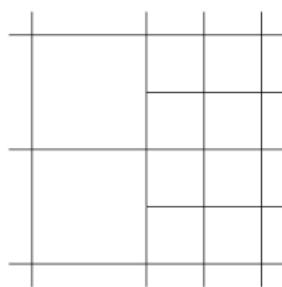


Dual mesh

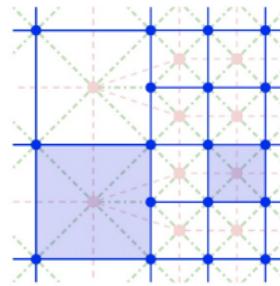


Diamond mesh

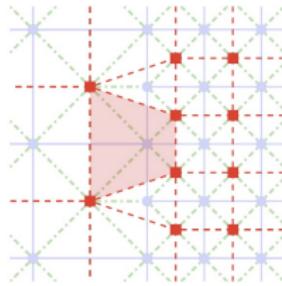
# Different kinds of DDFV meshes: composite



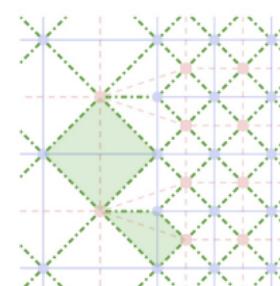
Initial mesh



Primal mesh



Dual mesh

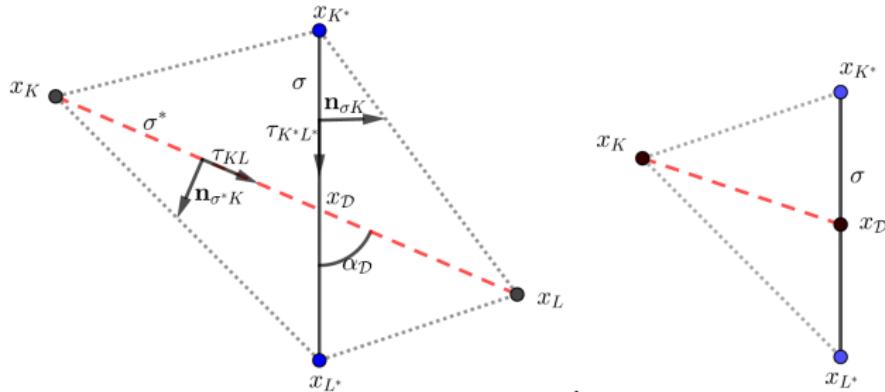


Diamond mesh

**Primal mesh associated to centers.** **Dual mesh associated to vertices.** **Diamond mesh associated to interfaces.**

# Discrete gradient by diamond

Hermeline 98, Coudière & al 99, Omnes & al 05



The discrete gradient is piecewise constant on the diamond cells

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{\sin(\alpha_{\mathcal{D}})} \left( \frac{u_L - u_K}{|\sigma^*|} \mathbf{n}_{\sigma K} + \frac{u_{L^*} - u_{K^*}}{|\sigma|} \mathbf{n}_{\sigma^* K^*} \right), \quad \forall \mathcal{D} \in \mathfrak{D}.$$

This definition has been invented in such a way that

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} \cdot \tau_{KL} = \frac{u_L - u_K}{|\sigma^*|}, \quad \nabla^{\mathcal{D}} u_{\mathcal{T}} \cdot \tau_{K^* L^*} = \frac{u_{L^*} - u_{K^*}}{|\sigma|}.$$

$$\|\nabla^{\mathcal{D}} u_{\mathcal{T}}\|_2^2 = \sum_{\mathcal{D} \in \mathfrak{D}} |\mathcal{D}| \|\nabla^{\mathcal{D}} u_{\mathcal{T}}\|_{\mathbb{R}^2}^2.$$

# DDFV discretization

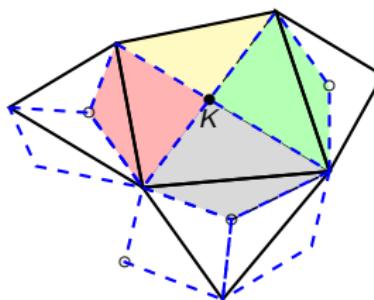
Integrating equation on the primal mesh and it is deduced similarly in the case of the dual mesh. So, let  $n \in \{0, \dots, N-1\}$  and  $K$  be a primal control volume. Then, one gets

$$\int_{t^n}^{t^{n+1}} \int_K \partial_t u \, dx \, dt - \sum_{\sigma \in \mathcal{E}_K} \int_{t^n}^{t^{n+1}} \int_\sigma a(u) \Lambda \nabla u \cdot \mathbf{n}_{\sigma K} \, d\sigma \, dt = 0. \quad (3.1)$$

The Kirchhoff function  $F(u)$  and the semi-Kirchhoff function  $\xi(u)$  are necessary for the

$$a(u) \nabla u = \nabla F(u), \quad a(u) \nabla u = v(u) \nabla \xi(u),$$

with  $v(u) = \sqrt{a(u)}$  and  $\xi'(u) = \sqrt{a(u)}$ .



Discrete gradient Diamond associated with edges  $\sigma_K \in \mathcal{E}$ .

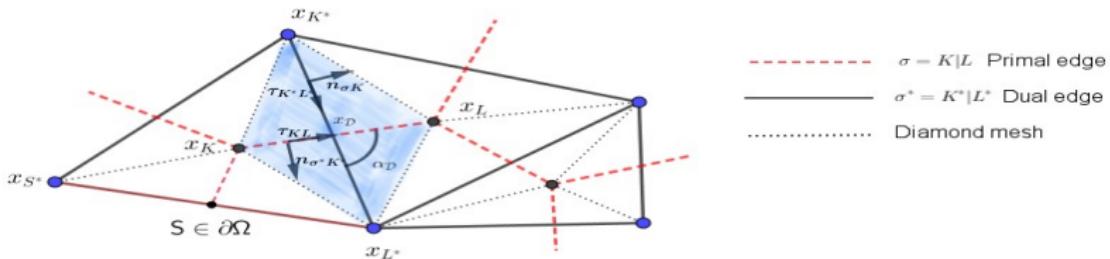
$$\int_K \operatorname{div}(\Lambda v(u) \nabla \xi(u)) \, dx \approx \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_K} \int_\sigma v(u) \Lambda \nabla^{\mathcal{D}} \xi(u) \cdot \mathbf{n}_{\sigma K} \, d\sigma,$$

# DDFV discretization

$$\begin{aligned}
 & \int_K \operatorname{div}(\Lambda v(u) \nabla \xi(u)) \, dx \\
 & \approx - \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_K} \frac{1}{\sin(\alpha_{\mathcal{D}})} \left( \frac{|\sigma|}{|\sigma^*|} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma K} \widehat{\nu}_{KL}(\xi(u_K) - \xi(u_L)) \right. \\
 & \quad \left. + \nu_{KL} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma^* K^*} (\xi(u_{K^*}) - \xi(u_{L^*})) \right), \\
 & = - \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_K} \left( \widehat{\nu}_{KL} \underbrace{a_{KL}}_{\geq 0} (\xi_K - \xi_L) + \nu_{KL} \underbrace{\eta_{\sigma \sigma^*}^{\mathcal{D}}}_{\in \mathbb{R}} (\xi_{K^*} - \xi_{L^*}) \right),
 \end{aligned}$$

where

$$a_{KL} := \frac{1}{\sin(\alpha_{\mathcal{D}})} \frac{|\sigma|}{|\sigma^*|} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma K} > 0, \quad \eta_{\sigma \sigma^*}^{\mathcal{D}} := \frac{1}{\sin(\alpha_{\mathcal{D}})} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma^* K^*} \in \mathbb{R}$$



# DDFV discretization

Approximation of fluxes on interfaces:

$$\int_K \nabla \cdot (v(u) \Lambda \nabla \xi(u)) dx \approx - \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_K} \left( \widehat{v}_{KL} a_{KL} (\xi_K - \xi_L) + v_{KL} \eta_{\sigma \sigma^*}^{\mathcal{D}} (\xi_{K^*} - \xi_{L^*}) \right)$$

where  $\widehat{v}_{KL}$  and  $v_{KL}$  are two different approximations of  $v(u)$  on  $\sigma$

$$\widehat{v}_{KL} = \begin{cases} \frac{F_K - F_L}{\xi_K - \xi_L} & \text{if } \xi_K \neq \xi_L, \\ v(u_K) & \text{else} \end{cases}, \quad (\text{centered})$$

and

$$v_{KL} = \begin{cases} v_{\downarrow}(u_L) + v_{\uparrow}(u_K) & \text{if } \eta_{\sigma \sigma^*}^{\mathcal{D}} (\xi_{K^*} - \xi_{L^*}) \geq 0, \\ v_{\downarrow}(u_K) + v_{\uparrow}(u_L) & \text{else} \end{cases}, \quad (\text{upwind})$$

The functions  $v_{\downarrow}, v_{\uparrow}$  are given by

$$v_{\uparrow}(u) := \int_0^u \max(v'(s), 0) ds, \quad v_{\downarrow}(u) := \int_0^u \min(v'(s), 0) ds.$$

The crucial choices of  $v_{KL}$  and  $\widehat{v}_{KL}$  will allow to prove the maximum principle and to bound the discrete gradient.

Similarly, one gets the approximations of the fluxes in the case  $K^* \in \mathfrak{M}^*$ .

# Nonlinear positive DDFV scheme

Using the implicit Euler scheme in time, one gets

$$u_M^0 = \frac{1}{|M|} \int_M u^0(x) dx, \quad \forall M \in \mathcal{T}, \quad (3.1)$$

$$\begin{aligned} & \frac{|K|}{\delta t} (u_K^{n+1} - u_K^n) \\ & + \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_K} \left( a_{KL} (F_K^{n+1} - F_L^{n+1}) + v_{KL} \eta_{\sigma \sigma^*}^{\mathcal{D}} (\xi_{K^*} - \xi_{L^*}) \right) \\ & + \mathcal{P}_K u_{\mathcal{T}}^{n+1} = 0, \quad \forall K \in \mathfrak{M},, \quad n \geq 0, \end{aligned} \quad (3.2)$$

$$\begin{aligned} & \frac{|K^*|}{\delta t} (u_{K^*}^{n+1} - u_{K^*}^n) \\ & + \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_{K^*}} \left( a_{K^* L^*} (F_{K^*}^{n+1} - F_{L^*}^{n+1}) + v_{KL} \eta_{\sigma \sigma^*}^{\mathcal{D}} (\xi_K - \xi_L) \right) \\ & + \mathcal{P}_{K^*} u_{\mathcal{T}}^{n+1} = 0, \quad \forall K \in \overline{\mathfrak{M}^*}, \quad n \geq 0. \end{aligned} \quad (3.3)$$

with  $\mathcal{P}_K u_{\mathcal{T}}^{n+1}$ : the penalization term will ensure the convergence of  $u_{\mathfrak{M}}$  and  $u_{\overline{\mathfrak{M}^*}}$  to the same limit.

# Maximum principle and energy estimate

## Lemma (Maximum principle)

For each fixed integer  $0 \leq n \leq N - 1$ , let  $(u_{\tau}^{n+1})$  be a sequence such that the DDFV scheme (3.2)-(3.3) holds. If  $u_{\mathfrak{M}}^0, u_{\mathfrak{M}^*}^0$  belong to  $[0, 1]$  then  $u_{\mathfrak{M}}^{n+1}, u_{\mathfrak{M}^*}^{n+1}$  remain also in  $[0, 1]$ .

## Proposition (Energy estimate)

Let  $(u_{\tau}^{n+1})_{n=0, \dots, N-1}$  such that the DDFV scheme (3.1)-(3.3) holds. Then,

$$\sum_{n=0}^{N-1} \delta t \|\nabla^{\mathfrak{D}} \xi_{\tau}^{n+1}\|_2^2 + \frac{1}{2} \frac{1}{h_{\mathfrak{D}}^{\varepsilon}} \sum_{n=0}^{N-1} \delta t \|\xi(u_{\mathfrak{M}}^{n+1}) - \xi(u_{\mathfrak{M}^*}^{n+1})\|_{L^2(\Omega)}^2 \leq C,$$

$$\sum_{n=0}^{N-1} \delta t \|\nabla^{\mathfrak{D}} F_{\tau}^{n+1}\|_2^2 \leq C.$$

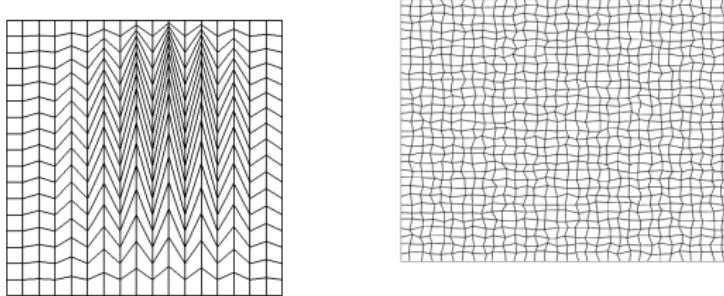


E.H. Quenjel, M. Saad, M. Ghilani, M. Bessemoulin-Chatard, *Convergence of a positive nonlinear DDFV scheme for degenerate parabolic equations*, Calcolo 57, 19 (2020).



M. Ibrahim, E.H. Quenjel, M. Saad, *Positive nonlinear DDFV scheme for a degenerate parabolic system describing chemotaxis*, CAMWA, (2020) 2972–3003

## Data



Kershaw quadrangle and quadrangle meshes,

The considered medium is  $\Omega = [0, 1]^2$ . The mobility function is chosen to be  $a(u) = u^m(1 - u)^m$ ,  $m \in \{1, 2\}$ . Therefore

$$v_{\uparrow}(u) = v\left(\min\left\{u, \frac{1}{2}\right\}\right), \quad \text{and} \quad v_{\downarrow}(u) = v\left(\max\left\{u, \frac{1}{2}\right\}\right) - v\left(\frac{1}{2}\right), \quad \text{for all } u \in (0, 1)^2.$$

The matrix  $\Lambda$  is given by

$$\Lambda = \begin{pmatrix} \Lambda_{xx} & 0 \\ 0 & \Lambda_{yy} \end{pmatrix}.$$

To evaluate the error, we denote

$$EL2 = \|u_{\text{ex}} - u_{\mathcal{T}, \delta_t}\|_{L^\infty(0, T; L^2(\Omega))}, \quad EGL2 = \|\nabla \xi(u_{\text{ex}}) - \nabla \xi(u_{\mathcal{T}, \delta_t})\|_{L^2(\Omega \times (0, T))^2}.$$

## Test 1

$$u_{\text{ex}}(x, y, t) = 6x^2 \times t, \quad \in \Omega, \quad t \in (0, 0.15).$$

Kershaw mesh						
h	$\ u_{\text{ex}} - u_{\text{app}}\ _{L^\infty(L^2)}$	Rate	$\ \nabla \xi(u_{\text{ex}}) - \nabla \xi(u_{\text{app}})\ _{L^2}$	Rate	$u_{\min}$	$u_{\max}$
0.342	0.104 E-01	-	0.367 E-01	-	0	0.840
0.174	0.425 E-02	1.335	0.242 E-01	0.622	0	0.895
0.092	0.132 E-02	1.821	0.138 E-01	0.878	0	0.897
0.047	0.365 E-03	1.933	0.696 E-02	1.026	0	0.900
0.0195	0.114 E-03	1.312	0.385 E-02	0.667	0	0.900

Quadrangle irregular mesh						
h	$\ u_{\text{ex}} - u_{\text{app}}\ _{L^\infty(L^2)}$	Rate	$\ \nabla \xi(u_{\text{ex}}) - \nabla \xi(u_{\text{app}})\ _{L^2}$	Rate	$u_{\min}$	$u_{\max}$
0.2710	0.124 E-02	-	0.102 E-00	-	0	0.882
0.1355	0.365 E-03	1.767	0.357 E-01	1.519	0	0.892
0.0903	0.173 E-03	1.849	0.212 E-01	1.286	0	0.897
0.0677	0.100 E-03	1.890	0.151 E-01	1.171	0	0.898
0.0542	0.654 E-04	1.914	0.118 E-01	1.111	0	0.900

Tabelle: Numerical convergence with  $\Lambda_{xx} = \Lambda_{yy} = 1$  and  $a(u) = u(1 - u)$ .

## Test 1

$$u_{\text{ex}}(x, y, t) = 6x^2 \times t, \quad \in \Omega, \quad t \in (0, 0.15).$$

Kershaw mesh						
h	ERL2	Rate	ERGL2	Rate	u <sub>min</sub>	u <sub>max</sub>
0.342	0.116 E-01	-	0.380 E-01	-	0	0.840
0.174	0.506 E-02	1.245	0.263 E-01	0.547	0	0.895
0.092	0.199 E-02	1.459	0.173 E-01	0.659	0	0.897
0.047	0.754 E-03	1.453	0.108 E-01	0.703	0	0.900
0.0195	0.207 E-03	1.459	0.580 E-02	0.701	0	0.900

Quadrangle irregular mesh						
h	ERL2	Rate	ERGL2	Rate	u <sub>min</sub>	u <sub>max</sub>
0.2710	0.210 E-02	-	0.118 E-00	-	0	0.881
0.1355	0.672 E-03	1.646	0.533 E-01	1.148	0	0.893
0.0903	0.326 E-03	1.783	0.359 E-01	0.974	0	0.897
0.0677	0.193 E-03	1.833	0.272 E-01	0.961	0	0.898
0.0542	0.127 E-03	1.852	0.220 E-01	0.966	0	0.900

Tabelle: Numerical convergence with  $\Lambda_{xx} = 1$ ,  $\Lambda_{yy} = 0.001$  and  $a(u) = u(1 - u)$ .

## Test 2

An example which exhibits a low space regularity due to the degenerate nature of the considered

$$u_{\text{ex}}(x, y, t) = \max(2\Lambda_{xx}t - x, 0) \quad \in \Omega, t \in (0, T),$$

Kershaw mesh					
h	ERL2	Rate	ERGL2	Rate	u <sub>min</sub>
0.342	0.340 E-01	-	0.103 E-00	-	0.734 E-03
0.174	0.123 E-01	1.516	0.621 E-01	0.752	0.131 E-03
0.092	0.336 E-02	2.022	0.335 E-01	0.963	0.178 E-04
0.047	0.847 E-03	2.068	0.168 E-01	1.040	0.449 E-05
0.0195	0.222 E-03	1.509	0.884 E-02	0.728	0.234 E-06

Quadrangle irregular mesh					
h	ERL2	Rate	ERGL2	Rate	u <sub>min</sub>
0.2710	0.286 E-02	-	0.251 E-00	-	0.130 E-05
0.1355	0.713 E-03	2.004	0.119 E-00	0.987	0.108 E-05
0.0903	0.317 E-03	2.000	0.794 E-01	0.991	0.563 E-06
0.0677	0.179 E-03	1.991	0.598 E-01	0.988	0.739 E-07
0.0542	0.115 E-03	1.990	0.479 E-01	0.990	0.172 E-08

Tabelle: Numerical convergence with  $\Lambda_{xx} = 1$ ,  $\Lambda_{yy} = 10$  and  $a(u) = 2u$

# Conclusion

- ① DDFV for compressible and incompressible two phase flows
- ② Positive CVFE scheme with  $\mathbb{P}_k$ ,  $k \geq 2$ ?

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 ありがとうございます