Reliable, Robust, & Efficient A Posteriori Estimates for Nonlinear Elliptic Problems using Linearization

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0 Outline

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Nonlinear elliptic problems

For $d \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^d$ be an open and bounded polytope. Let $u \in H_0^1(\Omega)$ solve the elliptic operator equation: for $\mathcal{R} : H_0^1(\Omega) \to H^{-1}(\Omega)$,

$$\langle \mathcal{R}(u), \varphi \rangle = 0, \quad \forall\, \varphi \in H_0^1(\Omega).$$

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Assumption 1 $\mathcal{R}$ is monotone & Lipschitz in some sense*

For an arbitrary $\tilde{u} \in H_0^1(\Omega)$, and constants $\lambda_{\mathrm{M}} > \lambda_{\mathrm{m}} > 0$

$$\lambda_{\mathrm{m}} \operatorname{dist}(\tilde{u}, u) \leq \sup_{\varphi \in H_0^1(\Omega)} \frac{\langle \mathcal{R}(\tilde{u}), \varphi \rangle}{\|\nabla \varphi\|} \leq \lambda_{\mathrm{M}} \operatorname{dist}(\tilde{u}, u)$$

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This implies that the error of $\tilde{u} \in H_0^1(\Omega)$ can simply be measured by

$$\|\mathcal{R}(\tilde{u})\|_{H^{-1}(\Omega)} := \sup_{\varphi \in H_0^1(\Omega)} \frac{\langle \mathcal{R}(\tilde{u}), \varphi \rangle}{\|\nabla \varphi\|}$$

1 Objective

- To have **reliable**, **locally efficient** a posteriori error estimates **robust with respect to nonlinearities**

$$C_{\mathrm{m}}\, \eta(\tilde{u}) \leq \|\mathcal{R}(\tilde{u})\|_{H^{-1}(\Omega)} \leq C_{\mathrm{M}}\, \eta(\tilde{u})$$

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$$C_{\mathrm{m}}\,\eta(\tilde{u}) \leq \|\mathcal{R}(\tilde{u})\|_{H^{-1}(\Omega)} \leq C_{\mathrm{M}}\,\eta(\tilde{u})$$

However, generally $C_{\mathrm{M}}/C_{\mathrm{m}}$ depends on $\lambda_{\mathrm{M}}/\lambda_{\mathrm{m}}$ which can be large, and thus the estimate is not robust with respect to nonlinearities

1 A linear example

Consider the diffusion eq: $\langle \mathcal{R}(u), \varphi \rangle := (A(\boldsymbol{x})\nabla u, \nabla \varphi) - (f, \varphi) = 0$.

Let $\lambda_{\mathrm{m}}^2 \leq A(\boldsymbol{x}) \leq \lambda_{\mathrm{M}}^2$.

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Let $\lambda_{\mathrm{m}}^2 \leq A(\boldsymbol{x}) \leq \lambda_{\mathrm{M}}^2$. If $u_h \in V_h \subset H_0^1(\Omega)$ is the f.e. solution of the problem then Cea's lemma gives

$$\|\nabla(u - u_h)\| \leq \frac{\lambda_{\mathrm{M}}}{\lambda_{\mathrm{m}}} \|\nabla(u_h - \varphi_h)\|, \quad \forall \varphi_h \in V_h.$$

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However, defining the energy norm $|||\varphi|||_{1,A} = \|A(\boldsymbol{x})^{\frac{1}{2}} \nabla \varphi\|$ one has

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Similarly, if we use the error measure

$$|||\mathcal{R}(\tilde{u})|||_{-1,A} := \sup_{\varphi \in H_0^1(\Omega)} \frac{\langle \mathcal{R}(\tilde{u}), \varphi \rangle}{\|\varphi\|_{1,A}}$$

then we have robust estimates [Repin (2000)]

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Then $|||\mathcal{R}(\cdot)|||_{-1,A(\cdot,u)}$ cannot be defined since $u \in H_0^1(\Omega)$ is unknown.

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Linearization iterations

We generally solve nonlinear equations by linearization iterations, i.e., by finding a sequence $\{u_h^i\}_{i \in \mathbb{N}} \subset V_h \subset H_0^1(\Omega)$.

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Example (Fixed point iteration) For each $i \in \mathbb{N}$ and $u_h^i \in V_h$, let $u_h^{i+1} \in V_h$ be the finite element solution of

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Then defining the energy norms generated at iteration i as

$$\begin{cases} |||\varphi|||_{1,u_h^i} := \|A(\boldsymbol{x}, u_h^i)^{\frac{1}{2}} \nabla \varphi\| & \text{for } \varphi \in H_0^1(\Omega), \\ |||\varsigma|||_{-1,u_h^i} := \sup_{\varphi \in H_0^1(\Omega)} \langle \varsigma, \varphi \rangle / |||\varphi|||_{1,u_h^i} & \text{for } \varsigma \in H^{-1}(\Omega), \end{cases}$$

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we have robust estimates of $\left|\left|\left| \mathcal{R}_{\mathrm{lin}}^{u_h^i}(u_h^{i+1}) \right|\right|\right|_{-1,u_h^i}$ (discretization error)

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we have robust estimates of $\left|\left|\left| \mathcal{R}_{\text{lin}}^{u_h^i}(u_h^{i+1}) \right|\right|\right|_{-1,u_h^i}$ (discretization error)

but not of $|||\mathcal{R}(u_h^{i+1})|||_{-1,u_h^i}$

1 Questions

- Can we get a robust estimate of $|||\mathcal{R}(u_h^i)|||_{-1,u_h^i}$ using the linearization iterations?

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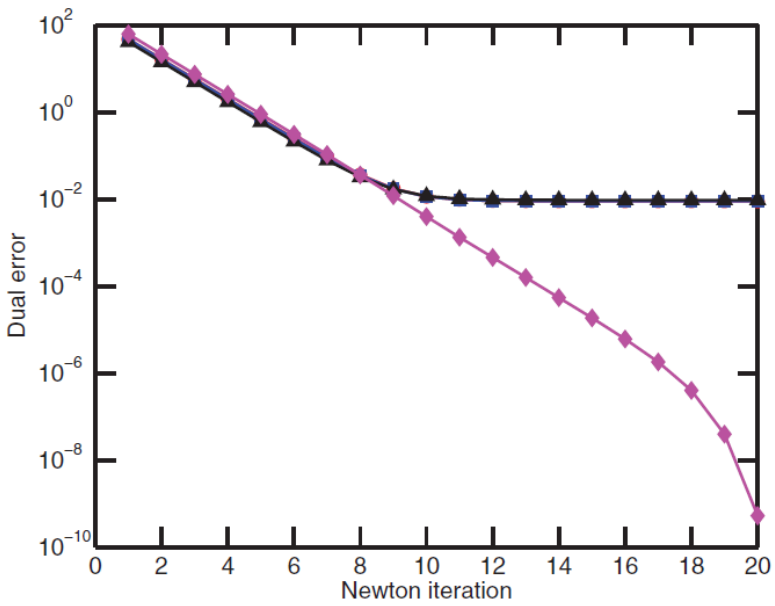
- If yes, then can this be used to stop the iterations adaptively?

[Ern & Vohralik (2013)]



See also: [El Alaoui *et al* (2011)], [Blechta *et al* (2018)], [Heid & Wihler (2020)]

2 Outline

Theorem 1 Decomposition of the total error using linearization

Under Assumption 1, and provided that the linearization iterations $\{u_h^i\}_{i\in\mathbb{N}} \subset V_h \subset H_0^1(\Omega)$ are generated by linear problems

$$\langle \mathcal{R}_{\mathrm{lin}}^{u_h^i}(u), \varphi\rangle = \mathfrak{L}(u_h^i; u - u_h^i, \varphi) + \langle \mathcal{R}(u_h^i), \varphi\rangle = 0, \quad \forall\, \varphi \in H_0^1(\Omega),$$

for a symmetric, bounded, coercive, bilinear form $\mathfrak{L}(u_h^i, \cdot, \cdot)$, and

$$|||\varphi|||_{1,u_h^i} = \mathfrak{L}(u_h^i; \varphi, \varphi)^{\frac{1}{2}}, \quad |||\varsigma|||_{-1,u_h^i} = \sup_{\varphi\in H_0^1(\Omega)} \frac{\langle \varsigma, \varphi\rangle}{|||\varphi|||_{1,u_h^i}},$$

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we have

$$\underbrace{|||\mathcal{R}(u_h^i)|||^2_{-1,u_h^i}}_{\text{total error}} = \underbrace{\left|\left|\left|\mathcal{R}_{\mathrm{lin}}^{u_h^i}(u_h^{i+1})\right|\right|\right|^2_{-1,u_h^i}}_{\text{discretization error of the linerization step}} + \underbrace{|||u_h^{i+1} - u_h^i|||^2_{1,u_h^i}}_{\text{linearization error}}.$$

- The linerization error is computed directly, we define

$$\eta^i_{\mathrm{lin},\Omega} := \left|\left|\left| u_h^{i+1} - u_h^i \right|\right|\right|^2_{1,u_h^i}.$$

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- For estimating $\left|\left|\left| \mathcal{R}^{u_h^i}_{\mathrm{lin}}(u_h^{i+1}) \right|\right|\right|^2_{-1,u_h^i}$ we introduce $\eta^i_{\mathrm{disc},\Omega}$, following the analysis on robust estimates of singularly perturbed reaction-diffusion problems in [Smears & Vohralík (2020)], [Verfürth (1998), (2005)]

Theorem 2 Reliable, efficient, and robust a posteriori estimates

Global reliability

$$|||\mathcal{R}(u^i_{\mathcal{T}})|||^2_{-1,u^i_h} \leq [\eta^i_\Omega]^2 = \sum_{K\in\mathcal{T}} ([\eta^i_{\mathrm{disc},K}]^2 + [\eta^i_{\mathrm{lin},K}]^2).$$

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Global efficiency

$$[\eta^i_\Omega]^2 \lesssim |||\mathcal{R}(u^i_h)|||^2_{-1,u^i_h} + \text{(data oscillation terms)}.$$

Local efficiency

For $\omega \subset \Omega$, there exists a neighbourhood $\mathfrak{T}_\omega \subseteq \Omega$ such that

$$[\eta^i_\omega]^2 \lesssim |||\mathcal{R}(u^{i+1}_h)|||^2_{-1,u^i_h,\mathfrak{T}_\omega} + [\eta^i_{\mathrm{lin},\mathfrak{T}_\omega}]^2 + \text{(data oscillation terms)}.$$

3 Outline

Class 1: gradient independent diffusivity problems

For all $\varphi \in H_0^1(\Omega)$, $\mathcal{R} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined as

$$\langle \mathcal{R}(u), \varphi \rangle := \langle f(\boldsymbol{x}, u), \varphi \rangle + \tau(\bar{\mathbf{K}}(\boldsymbol{x})(\mathcal{D}(\boldsymbol{x}, u)\nabla u + \boldsymbol{q}(\boldsymbol{x}, u)), \nabla \varphi)$$

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Semilinear equations $\Delta u = f(\boldsymbol{x}, u)$

Such equations pop up in quantum mechanics (special solutions to nonlinear Klein-Gordon equations), gravitation influences on stars, membrane buckling problems etc.

Time-discrete nonlinear advection-reaction-diffusion equations

with time-step $\tau > 0$ following evolutions equations reduce to this case

Poro-Fischer equations:	$\partial_t u = \Delta u^m + \lambda\, u\,(1-u)$
Richards equation:	$\partial_t S(u) = \nabla \cdot [\bar{\mathbf{K}}(\boldsymbol{x})\kappa(S(u))(\nabla u + \boldsymbol{g})] + f(\boldsymbol{x}, u)$
Biofilm equations:	$\partial_t u_k = \mu_k \Delta \Phi_k(u_k) + f_k((u_k)_{k=1}^n)$

3 Class of problems

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Assumption 1 is satisfied if $\tau > 0$ is small, and

- $\mathcal{D} : \Omega \times \mathbb{R} \to \mathbb{R}^+$ is bounded and Lipschitz
- $\bar{\mathbf{K}} : \Omega \to \mathbb{R}^{d \times d}$ is symmetric positive definite
- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is monotone and Lipschitz upto the boundary
- $\boldsymbol{q} : \Omega \times \mathbb{R} \to \mathbb{R}^d$ is bounded, and satisfies a Lipschitz condition*

with

$$\mathrm{dist}(u, v) = \left\| \bar{\mathbf{K}}^{\frac{1}{2}} \nabla \int_u^v \mathcal{D} \right\|$$

Class 2: gradient dependent diffusivity problems

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For $a(\cdot)$ satisfying the ellipticity condition, and $b(\cdot) > 0$	
Mean curvature flow:	$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y}) = \left(a(\boldsymbol{x}) + \frac{b(\boldsymbol{x})}{\sqrt{1+\lvert\boldsymbol{y}\rvert^2}} \right) \boldsymbol{y}$
p-Laplacian problems*:	$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y}) = (a(\boldsymbol{x}) + b(\boldsymbol{x})\lvert\boldsymbol{y}\rvert^{p-2})\boldsymbol{y}$
Compressive flow*:	$\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y}) = b(\boldsymbol{x}) \left(1 - \frac{r-1}{2}\lvert\boldsymbol{y}\rvert^2\right)^{\frac{1}{r-1}} \boldsymbol{y}$

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$$(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{z})) \cdot (\boldsymbol{y} - \boldsymbol{z}) \geq \lambda_{\mathrm{m}} |\boldsymbol{y} - \boldsymbol{z}|^2 \quad \text{for } \boldsymbol{x} \in \Omega \text{ and } \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^d,$$

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$$\mathrm{dist}(u, v) = \|\nabla(u - v)\|$$

Abstract linearization

For all $u_h^i \in V_h$, define the symmetric, coercive, and bounded bilinear form

$$\mathfrak{L}(u_h^i; v, w) := (L(\boldsymbol{x}, u_h^i)\, v, w) + (\mathfrak{a}(\boldsymbol{x}, u_h^i)\nabla v, \nabla w).$$

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With respect to $\mathfrak{L}$, the linearized energy norms are defined as

$$|||\varphi|||_{1,u_h^i} = \mathfrak{L}(u_h^i; \varphi, \varphi)^{\frac{1}{2}} = \left(\int_\Omega L(\boldsymbol{x}, u_h^i)\varphi^2 + |\mathfrak{a}(\boldsymbol{x}, u_h^i)^{\frac{1}{2}} \nabla \varphi|^2 \right)^{\frac{1}{2}},$$

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Scheme	$L(\boldsymbol{x}, v)$	$\mathfrak{a}(\boldsymbol{x}, v)/\tau$
Picard	$\partial_\xi f(\boldsymbol{x}, v)$	$\bar{\mathbf{K}}(\boldsymbol{x})\, \mathcal{D}(\boldsymbol{x}, v)$
Jäger–Kačur	$\max_{\xi \in \mathbb{R}} \left(\frac{f(\boldsymbol{x},\xi) - f(\boldsymbol{x},v)}{\xi - v} \right)$	$\bar{\mathbf{K}}(\boldsymbol{x})\, \mathcal{D}(\boldsymbol{x}, v)$
L-scheme	L (constant) $\geq \frac{1}{2} \sup \partial_\xi f$	$\bar{\mathbf{K}}(\boldsymbol{x})\, \mathcal{D}(\boldsymbol{x}, v)$
M-scheme	$\partial_\xi f(\boldsymbol{x}, v) + M\tau$ (constant)	$\bar{\mathbf{K}}(\boldsymbol{x})\, \mathcal{D}(\boldsymbol{x}, v)$

Examples in gradient independent diffusivity case

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Scheme	$L(\boldsymbol{x}, v)$	$\mathfrak{a}(\boldsymbol{x}, v)/\tau$
Picard	$\partial_\xi f(\boldsymbol{x}, v)$	$\bar{\mathbf{K}}(\boldsymbol{x})\, \mathcal{D}(\boldsymbol{x}, v)$
Jäger–Kačur	$\max_{\xi \in \mathbb{R}} \left(\frac{f(\boldsymbol{x},\xi) - f(\boldsymbol{x},v)}{\xi - v} \right)$	$\bar{\mathbf{K}}(\boldsymbol{x})\, \mathcal{D}(\boldsymbol{x}, v)$
L-scheme	L (constant) $\geq \frac{1}{2} \sup \partial_\xi f$	$\bar{\mathbf{K}}(\boldsymbol{x})\, \mathcal{D}(\boldsymbol{x}, v)$
M-scheme	$\partial_\xi f(\boldsymbol{x}, v) + M\tau$ (constant)	$\bar{\mathbf{K}}(\boldsymbol{x})\, \mathcal{D}(\boldsymbol{x}, v)$

Examples in gradient independent diffusivity case

- Newton scheme leads to a non-symmetric $\mathfrak{L}$ and is treated separately

Abstract linearization

For $\mathfrak{L}(u_h^i; v, w) := (L(\boldsymbol{x}, u_h^i)\, v, w) + (\mathfrak{a}(\boldsymbol{x}, u_h^i)\nabla v, \nabla w)$,

compute $u_h^{i+1} \in V_h$ as the f.e. solution of the equation

$$\mathfrak{L}(u_h^i; u_h^{i+1} - u_h^i, \varphi) = -\langle \mathcal{R}(u_h^i), \varphi \rangle, \quad \forall \varphi \in H_0^1(\Omega).$$

Scheme	$L(\boldsymbol{x}, v)$	$\mathfrak{a}(\boldsymbol{x}, v)/\tau$
Kačanov	$\partial_\xi f(\boldsymbol{x}, v)$	$A(\boldsymbol{x}, \lvert\nabla v\rvert)$
Zarantonello	0	$\Lambda\,(\text{constant}) > 0$

Examples in gradient dependent diffusivity case

4 Outline

Algorithm 1 Adaptive linearization

For a fixed $0 < \mu \ll 1$, we iterate until for some $i = \bar{i} \in N$,

$$\eta_{\mathrm{lin},\Omega}^{\bar{i}} \leq \mu \, [\eta_{\Omega}^{\bar{i}}].$$

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Effectivity indices

Global effectivity index: $\text{Eff. Ind.} := \eta_{\Omega}^{i} / \left|\left|\left| \mathcal{R}(u_h^i) \right|\right|\right|_{-1,u_h^i}$

Local effectivity index: $(\text{Eff. Ind.})_K := \eta_K^i / \left|\left|\left| \mathcal{R}(u_h^i) \right|\right|\right|_{-1,u_h^i,K}, \quad \forall\, K \in \mathcal{T},$

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Mesh

Three mesh-levels used: $h = \frac{0.1}{\ell}$ where $\ell \in \{1, 2, 4\}$

4 Gradient independent diffusivity case: Richards equation

For $\Omega = (0,1) \times (0,1)$ we study

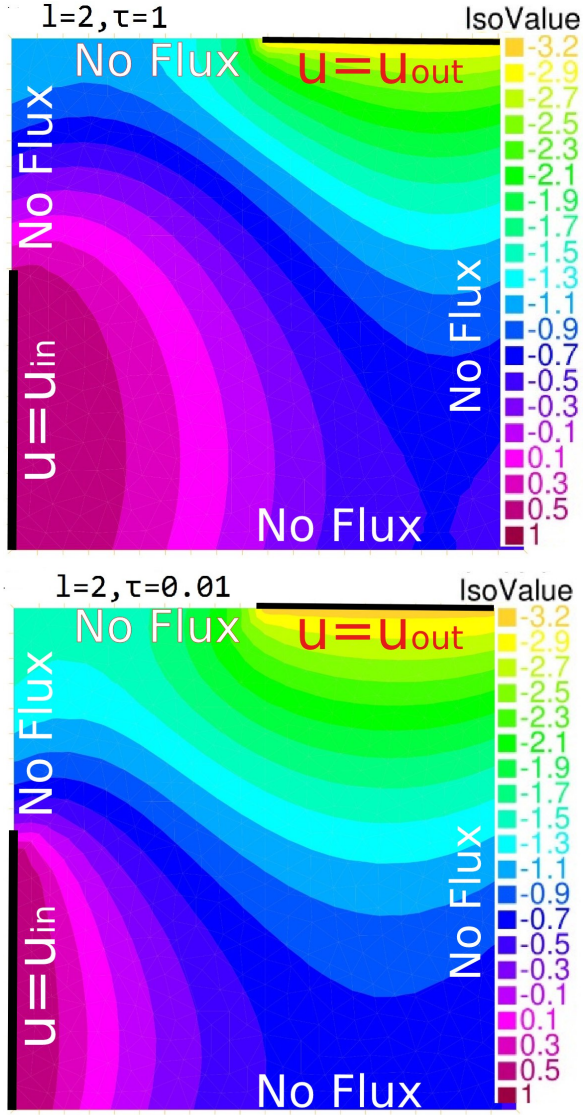
$$\langle \mathcal{R}(\tilde{u}), \varphi \rangle = (S(\tilde{u}) - S(\bar{u}), \varphi) + \tau(\bar{\mathbf{K}}\kappa(S(\tilde{u}))[\nabla \tilde{u} - \boldsymbol{g}], \nabla \varphi)$$

where the van Genuchten parametrization for S, κ is used:

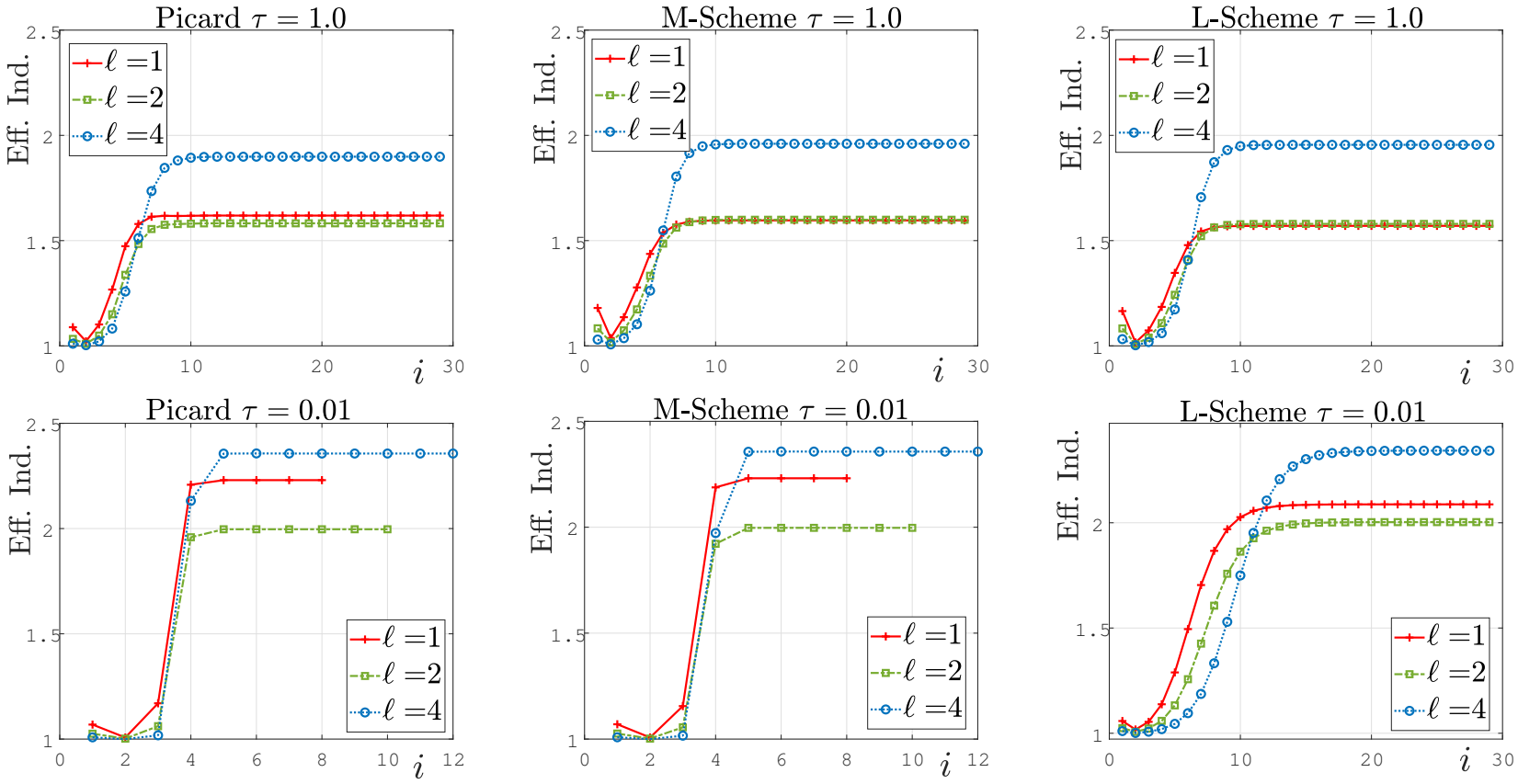
$$\begin{cases} S(\xi) := \left(1 + (2-\xi)^{\frac{1}{1-\lambda}}\right)^{-\lambda}, \\ \kappa(s) := \sqrt{s}\left(1 - (1 - s^{\frac{1}{\lambda}})^{\lambda}\right)^2, \end{cases}$$

with $\lambda = 0.5$, $u_h^0 = 0$,

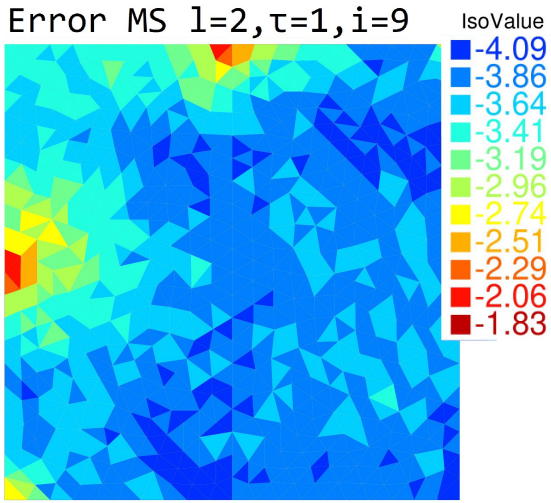
$$\bar{\mathbf{K}} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix}, \text{ and } \boldsymbol{g} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



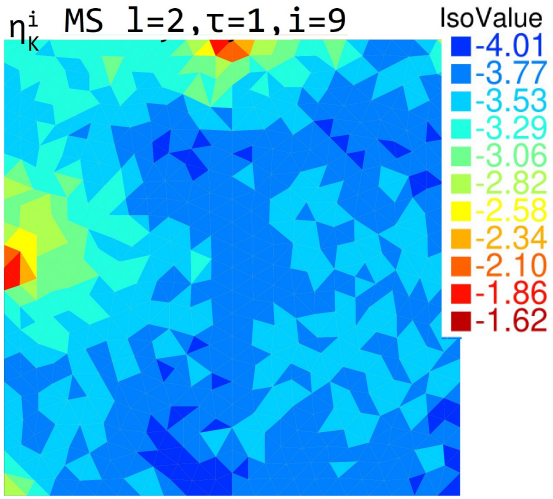
4 Global effectivity



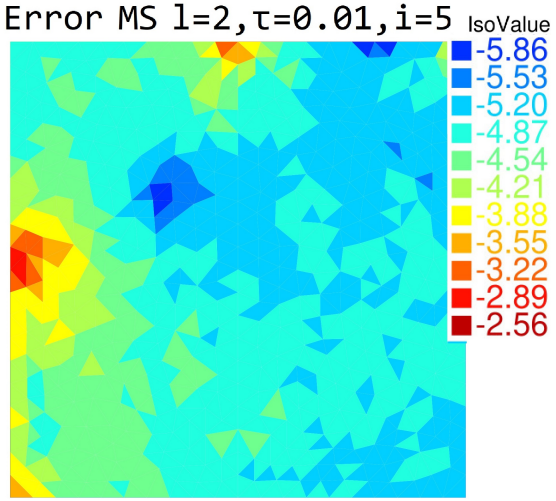
4 Distribution of error vs. estimates

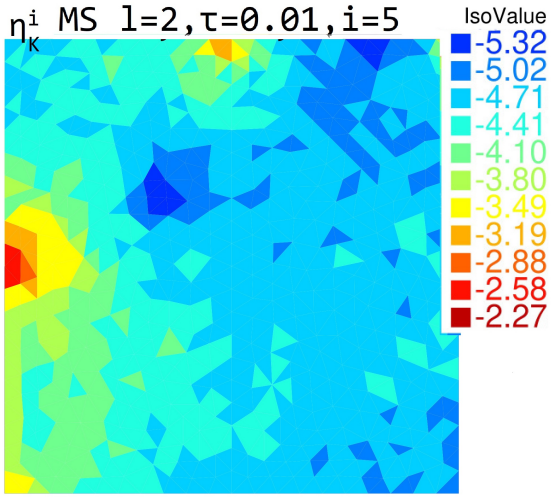


Error

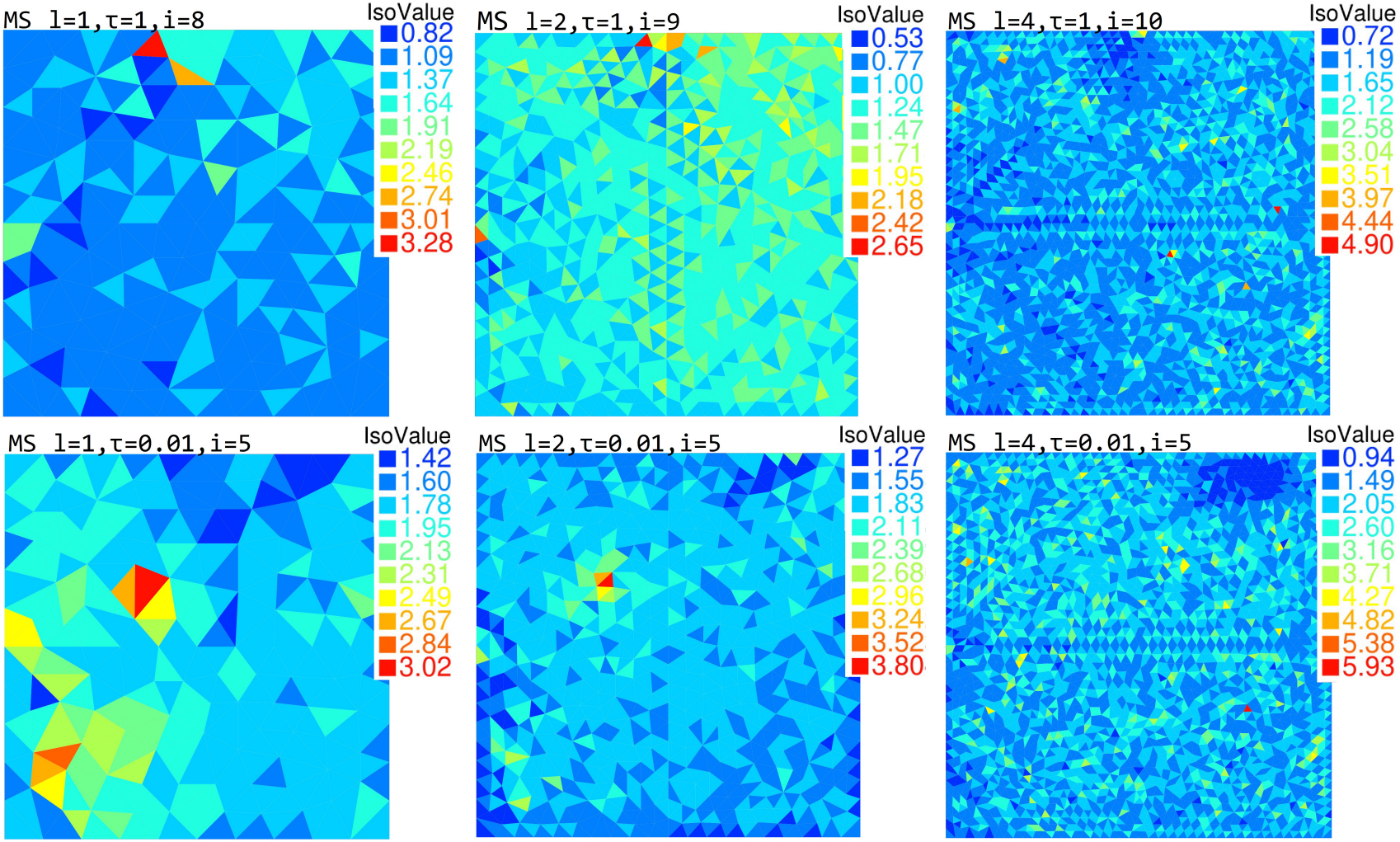


Estimate



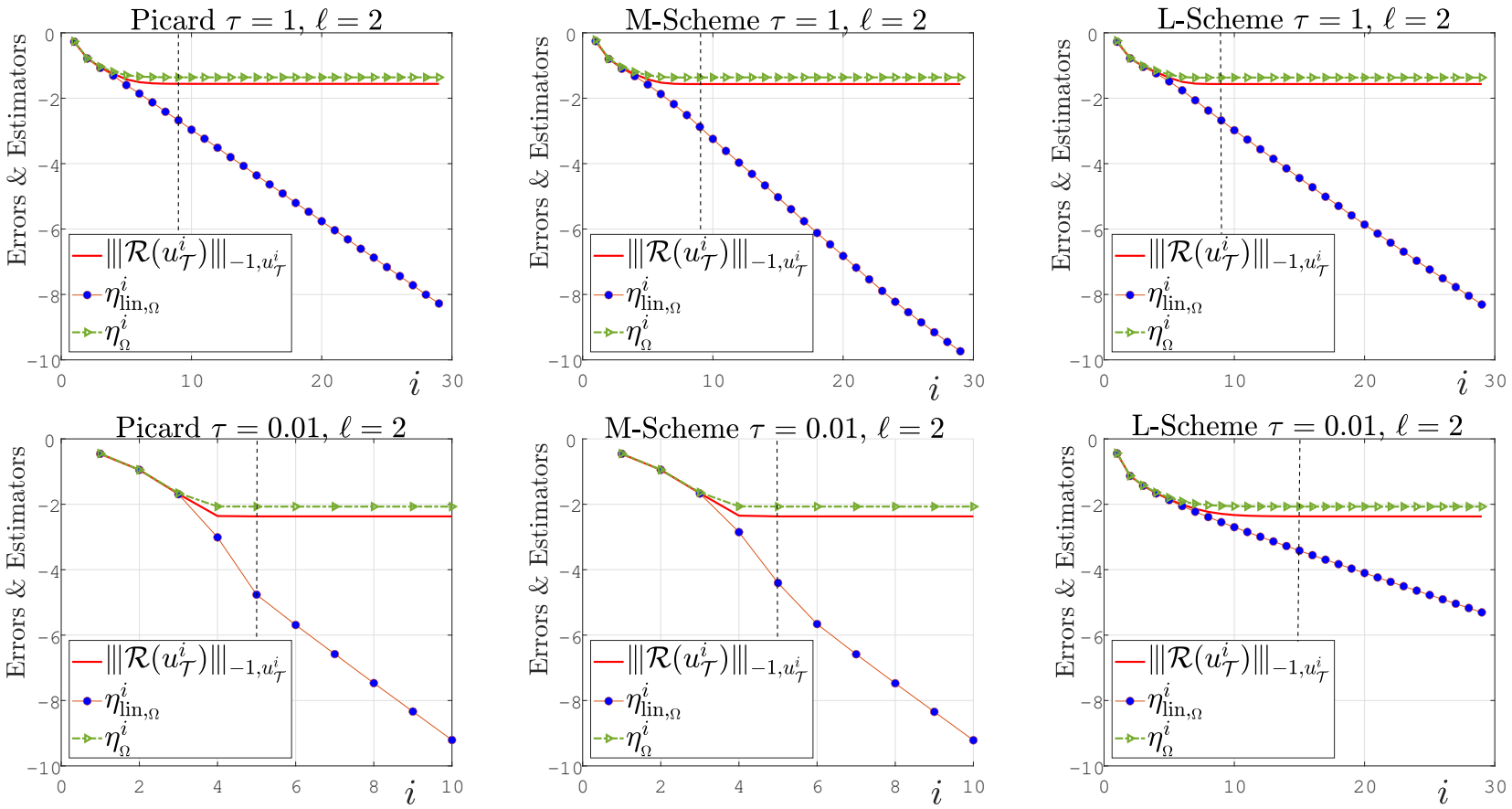


4 Local effectivity



4 Error with linearization iterations

For the adaptive stopping criteria $\mu = 0.05$ is chosen



4 Gradient independent diffusivity case

We consider in Ω the equation

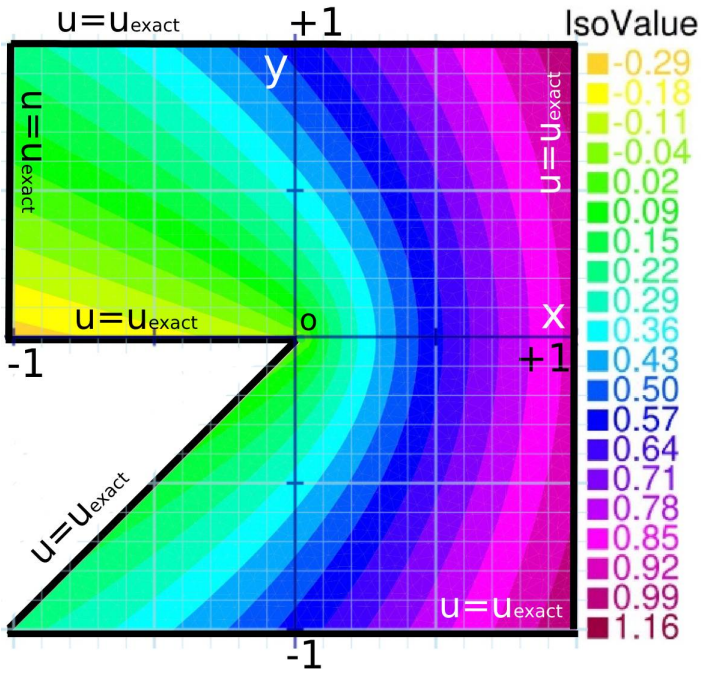
$$\varepsilon u - \nabla \cdot [A(|\nabla u|)\nabla u] = f$$

where

$$A(\boldsymbol{y}) = 2 + \frac{\boldsymbol{y}}{(1 + |\boldsymbol{y}|^2)},$$

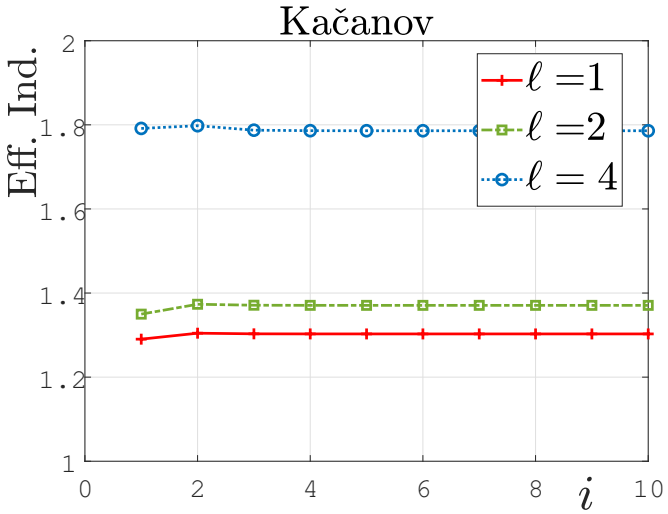
$\varepsilon = 10^{-2}$, and a singular $f \in H^{-1}(\Omega)$ is chosen such that the solution becomes

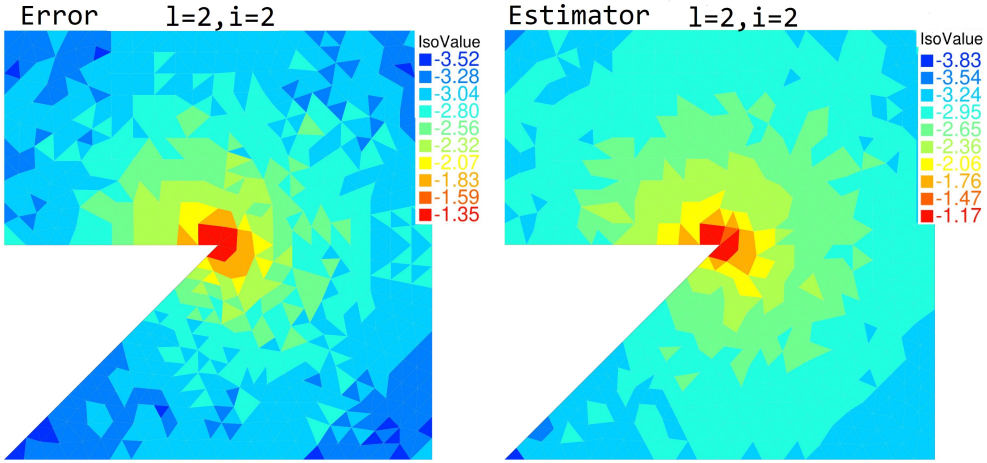
$$u_{\text{exact}} = r^{\frac{4}{7}} cos\left(\frac{4}{7}\theta\right).$$

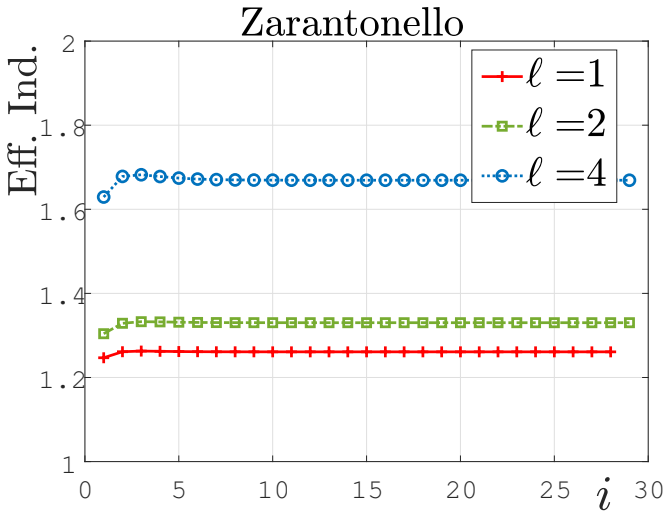


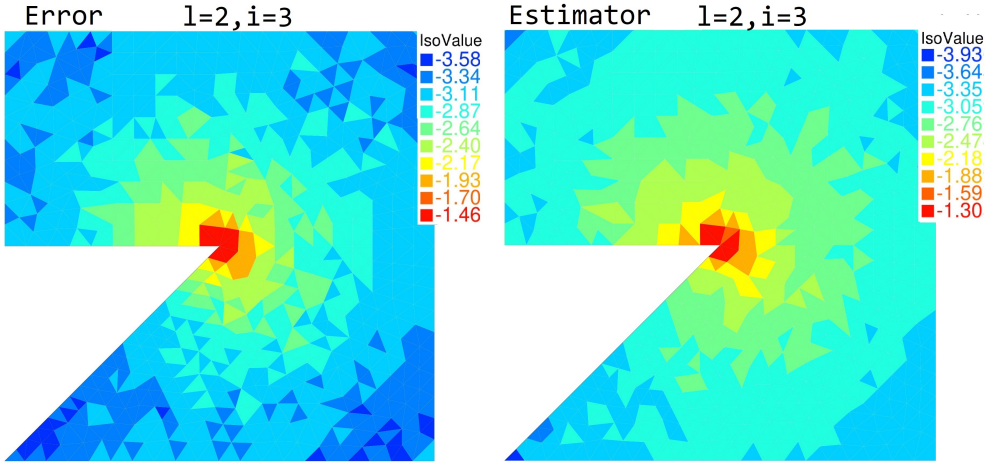
Ω

4 Global effectivity and distribution of error

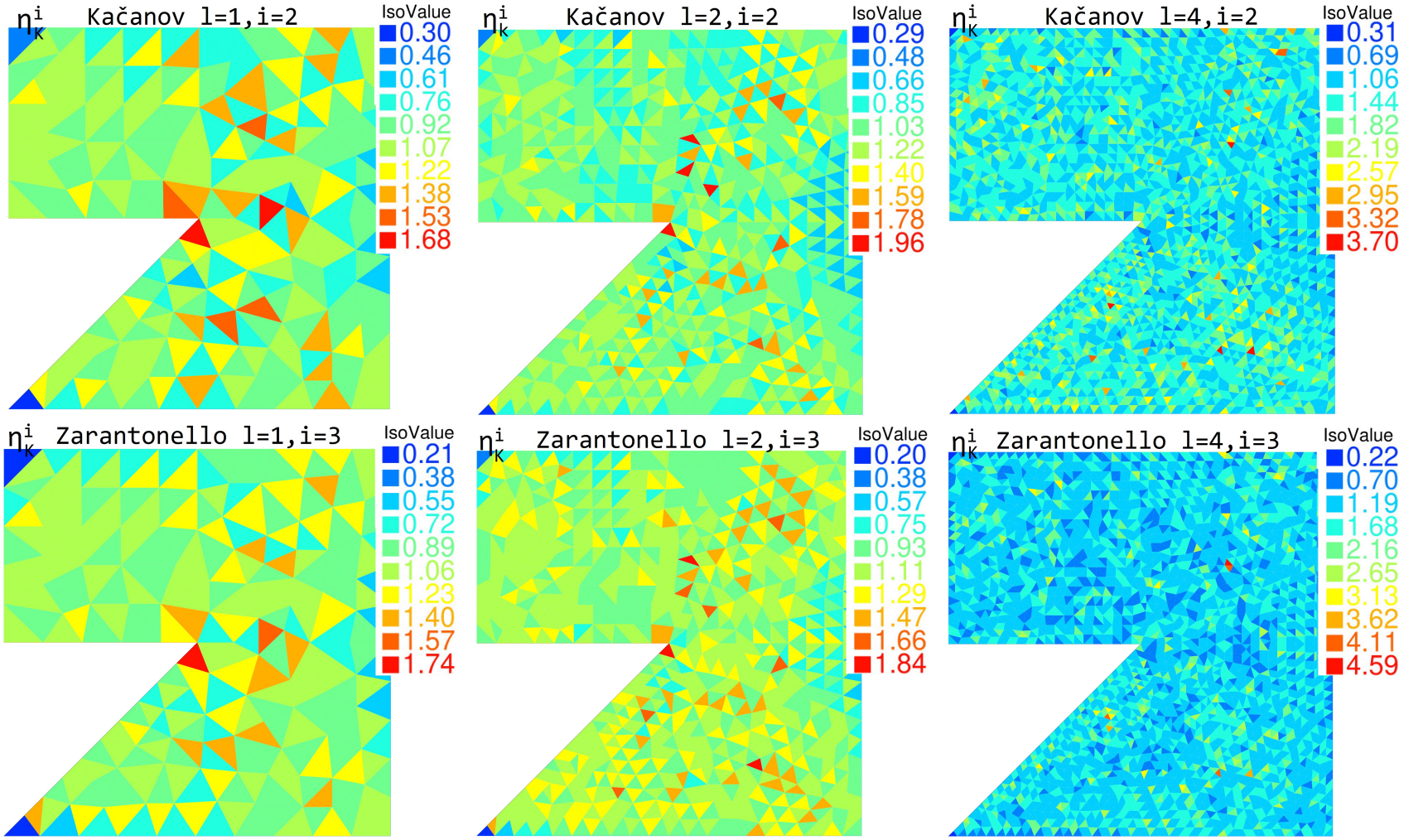




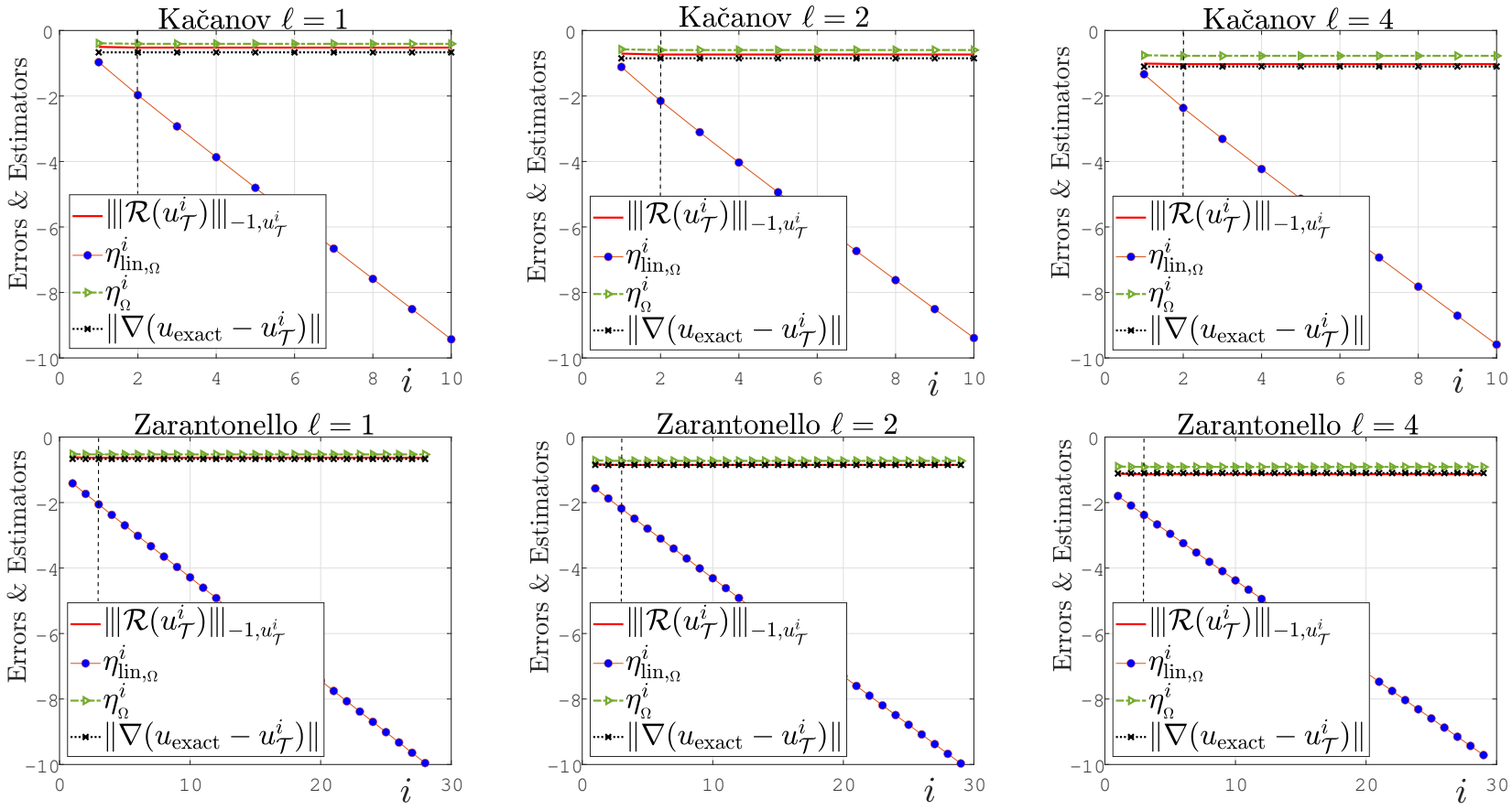




4 Local effectivity



4 Error with linearization iterations



4 The Newton scheme

For the Newton scheme, the linearization operator

$$\mathfrak{L}(u_h^i; v, w) := (L(\boldsymbol{x}, u_h^i)\, v, w) + (\mathfrak{a}(\boldsymbol{x}, u_h^i)\nabla v, \nabla w) + (\boldsymbol{w}(\boldsymbol{x}, u_h^i) v, \nabla w),$$

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is non-symmetric. However, if for some $C_N \in [0, 2)$ we have

$$\boldsymbol{w}(\boldsymbol{x}, u_{\mathcal{T}}^i)\, \mathfrak{a}^{-1}(\boldsymbol{x}, u_{\mathcal{T}}^i)\, \boldsymbol{w}(\boldsymbol{x}, u_{\mathcal{T}}^i) \leq C_N^2\, L(\boldsymbol{x}, u_{\mathcal{T}}^i), \quad \forall\, \boldsymbol{x} \in \Omega, \text{ and } i \in \mathbb{N},$$

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then,

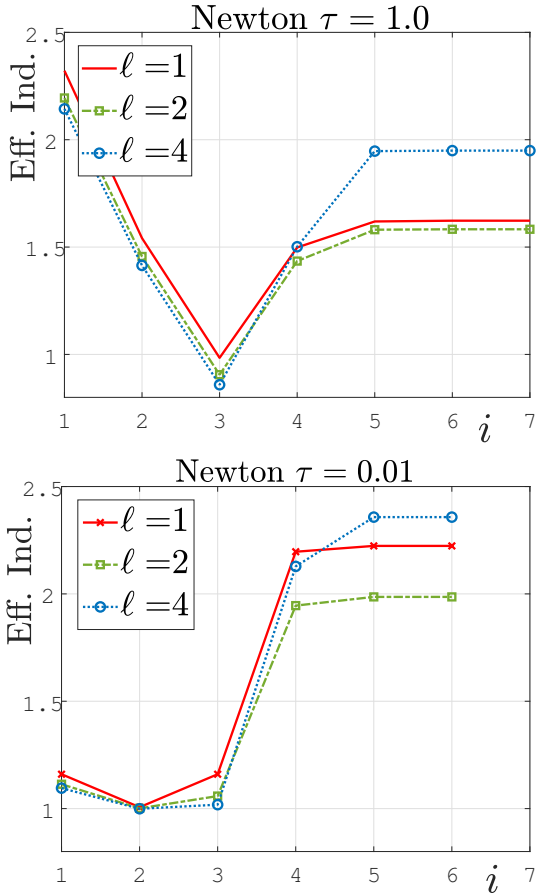
$$C_{\mathrm{m}}(C_N) \left[\left|\!\left|\!\left| \mathcal{R}_{\mathrm{lin}}^{u_h^i}(u_{\mathcal{T}}^{i+1}) \right|\!\right|\!\right|^2_{-1, u_h^i} + \left|\!\left|\!\left| u_{\mathcal{T}}^{i+1} - u_h^i \right|\!\right|\!\right|^2_{1, u_h^i} \right] \leq \left|\!\left|\!\left| \mathcal{R}(u_h^i) \right|\!\right|\!\right|^2_{-1, u_h^i}$$

$$\leq C_{\mathrm{M}}(C_N) \left[\left|\!\left|\!\left| \mathcal{R}_{\mathrm{lin}}^{u_h^i}(u_{\mathcal{T}}^{i+1}) \right|\!\right|\!\right|^2_{-1, u_h^i} + \left|\!\left|\!\left| u_{\mathcal{T}}^{i+1} - u_h^i \right|\!\right|\!\right|^2_{1, u_h^i} \right]$$

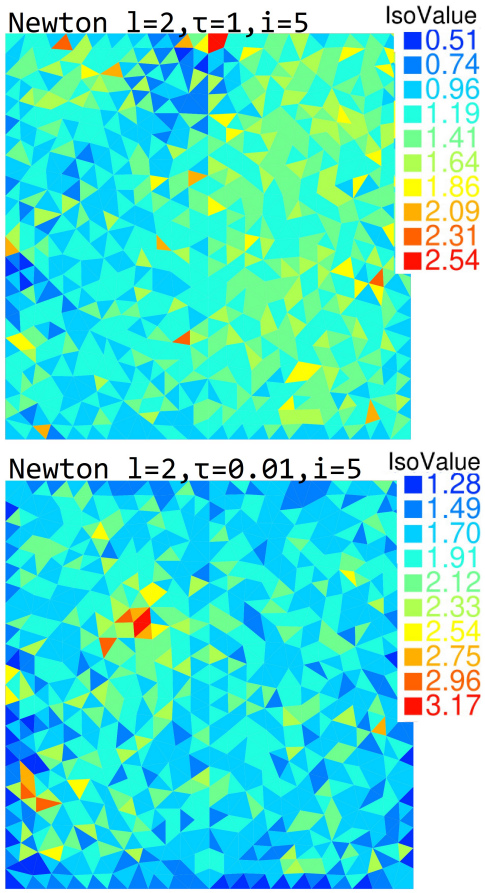
with $C_{\mathrm{m}}(C_N), C_{\mathrm{M}}(C_N) \to 1$ if $C_N \searrow 0$.

4 The Newton scheme: numerical results

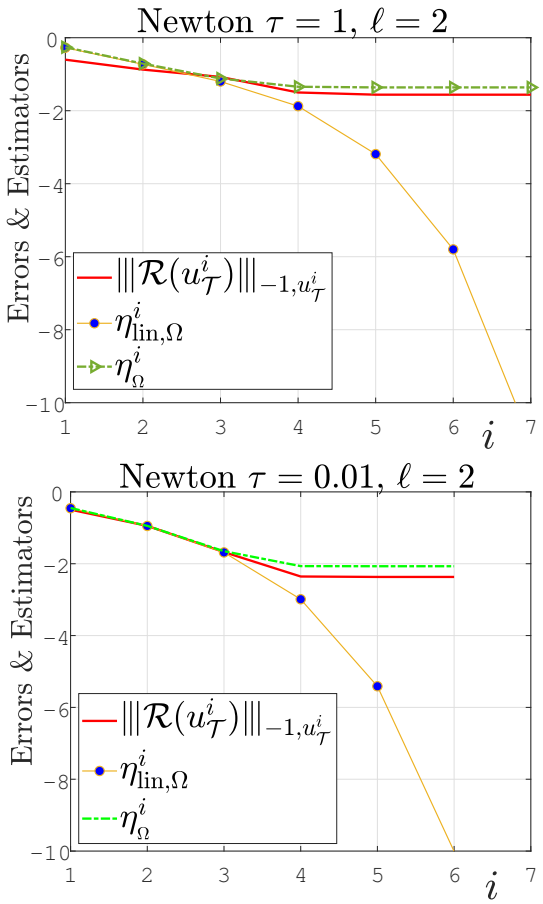
For gradient independent diffusivity case, we have



Global Effectivity



Local Effectivity



Error with iterations

4 Thank you for your time