Reliable, Robust, & Efficient A Posteriori Estimates for Nonlinear Elliptic Problems using Linearization

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InterPore
June 2022
Outline

1. Introduction
2. Main analytical results
3. Scope of the results
4. Numerical results
1 Outline

1 Introduction

2 Main analytical results

3 Scope of the results

4 Numerical results
1 Introduction

Nonlinear elliptic problems

For $d \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^d$ be an open and bounded polytope. Let $u \in H^1_0(\Omega)$ solve the elliptic operator equation: for $\mathcal{R} : H^1_0(\Omega) \rightarrow H^{-1}(\Omega)$,

$$\langle \mathcal{R}(u), \varphi \rangle = 0, \quad \forall \varphi \in H^1_0(\Omega).$$
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Assumption 1 $\mathcal{R}$ is monotone & Lipschitz in some sense*

For an arbitrary $\tilde{u} \in H^1_0(\Omega)$, and constants $\lambda_M > \lambda_m > 0$

$$\lambda_m \text{dist}(\tilde{u}, u) \leq \sup_{\varphi \in H^1_0(\Omega)} \frac{\langle \mathcal{R}(\tilde{u}), \varphi \rangle}{\| \nabla \varphi \|} \leq \lambda_M \text{dist}(\tilde{u}, u)$$
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$$\lambda_m \text{dist}(\tilde{u}, u) \leq \sup_{\varphi \in H^1_0(\Omega)} \frac{\langle \mathcal{R}(\tilde{u}), \varphi \rangle}{\| \nabla \varphi \|} \leq \lambda_M \text{dist}(\tilde{u}, u)$$

This implies that the error of $\tilde{u} \in H^1_0(\Omega)$ can simply be measured by

$$\| \mathcal{R}(\tilde{u}) \|_{H^{-1}(\Omega)} := \sup_{\varphi \in H^1_0(\Omega)} \frac{\langle \mathcal{R}(\tilde{u}), \varphi \rangle}{\| \nabla \varphi \|}$$
1 Objective

- To have **reliable, locally efficient** a posteriori error estimates **robust** with respect to nonlinearities

\[ C_m \eta(\tilde{u}) \leq \| R(\tilde{u}) \|_{H^{-1}(\Omega)} \leq C_M \eta(\tilde{u}) \]
1 Objective

- To have **reliable, locally efficient** a posteriori error estimates **robust with respect to nonlinearities**

\[
C_m \eta(\tilde{u}) \leq \|R(\tilde{u})\|_{H^{-1}(\Omega)} \leq C_M \eta(\tilde{u})
\]

However, generally \(C_M/C_m\) depends on \(\lambda_M/\lambda_m\) which can be large, and thus the estimate is not robust with respect to nonlinearities.
Consider the diffusion eq: $\langle \mathcal{R}(u), \varphi \rangle := (A(x) \nabla u, \nabla \varphi) - (f, \varphi) = 0$.

Let $\lambda_m^2 \leq A(x) \leq \lambda_M^2$. 


Consider the diffusion eq: \[ \langle R(u), \varphi \rangle := (A(x) \nabla u, \nabla \varphi) - (f, \varphi) = 0. \]

Let \( \lambda_m^2 \leq A(x) \leq \lambda_M^2 \). If \( u_h \in V_h \subset H^1_0(\Omega) \) is the f.e. solution of the problem then Cea’s lemma gives

\[
\|
\nabla (u - u_h)\|
\leq \frac{\lambda_M}{\lambda_m} \|
\nabla (u_h - \varphi_h)\|, \quad \forall \varphi_h \in V_h.
\]
1 A linear example

Consider the diffusion eq: $\langle \mathcal{R}(u), \varphi \rangle := (A(x) \nabla u, \nabla \varphi) - (f, \varphi) = 0$.

Let $\lambda_m^2 \leq A(x) \leq \lambda_M^2$. If $u_h \in V_h \subset H_0^1(\Omega)$ is the f.e. solution of the problem then Cea’s lemma gives

$$\| \nabla (u - u_h) \| \leq \frac{\lambda_M}{\lambda_m} \| \nabla (u_h - \varphi_h) \|, \quad \forall \varphi_h \in V_h.$$

However, defining the energy norm $\| \varphi \|_{1,A} = \| A(x)^{\frac{1}{2}} \nabla \varphi \|$ one has

$$\| u - u_h \|_{1,A} \leq \| u_h - \varphi_h \|_{1,A}, \quad \forall \varphi_h \in V_h.$$
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\]

Similarly, if we use the error measure

\[
\| \mathcal{R}(\tilde{u}) \|_{-1,A} := \sup_{\varphi \in H^1_0(\Omega)} \frac{\langle \mathcal{R}(\tilde{u}), \varphi \rangle}{\| \varphi \|_{1,A}}
\]

then we have robust estimates [Repin (2000)]
1 Moving to the nonlinear case

Consider the nonlinear eq: $\langle \mathcal{R}(u), \varphi \rangle = (A(x, u) \nabla u, \nabla \varphi) - (f, \varphi) = 0$. 

Then $||\mathcal{R}(\cdot)||_{-1, A(\cdot, u)}$ cannot be defined since $u \in H^{1, 0}(\Omega)$ is unknown.
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Then \( |||\mathcal{R}(\cdot)|||_{1, A(\cdot, u)} \) cannot be defined since \( u \in H^1_0(\Omega) \) is unknown.
1 Moving to the nonlinear case

Consider the nonlinear eq: \( \langle R(u), \varphi \rangle = (A(x, u) \nabla u, \nabla \varphi) - (f, \varphi) = 0 \).

**Linearization iterations**

We generally solve nonlinear equations by linearization iterations, i.e., by finding a sequence \( \{u_h^i\}_{i \in \mathbb{N}} \subseteq V_h \subset H_0^1(\Omega) \).
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We generally solve nonlinear equations by linearization iterations, i.e., by finding a sequence $\{u_h^i\}_{i \in \mathbb{N}} \subset V_h \subset H^1_0(\Omega)$.

**Example** (Fixed point iteration) For each $i \in \mathbb{N}$ and $u_h^i \in V_h$, let $u_h^{i+1} \in V_h$ be the finite element solution of

$$\langle \mathcal{R}_{\text{lin}}(u_h^i), \varphi \rangle := (A(x, u_h^i) \nabla u, \nabla \varphi) - (f, \varphi) = 0$$
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\]

Then defining the energy norms generated at iteration \( i \) as

\[
\begin{align*}
\| \varphi \|_{1, u_h^i} & := \| A(x, u_h^i)^{\frac{1}{2}} \nabla \varphi \| \quad \text{for } \varphi \in H_0^1(\Omega), \\
\| \zeta \|_{-1, u_h^i} & := \sup_{\varphi \in H_0^1(\Omega)} \langle \zeta, \varphi \rangle / \| \varphi \|_{1, u_h^i} \quad \text{for } \zeta \in H^{-1}(\Omega),
\end{align*}
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1 Moving to the nonlinear case

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Then defining the energy norms generated at iteration \( i \) as

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\| \varphi \|_{1,u_h^i} := \| A(x, u_h^i)^{\frac{1}{2}} \nabla \varphi \|,
\]

\[
\| \varsigma \|_{-1,u_h^i} := \sup_{\varphi \in H^1_0(\Omega)} \langle \varsigma, \varphi \rangle / \| \varphi \|_{1,u_h^i}
\]

we have robust estimates of

\[\| \mathcal{R}_{\text{lin}}^{u_h^i}(u_h^{i+1}) \|_{-1,u_h^i} \] (discretization error)
1 Moving to the nonlinear case

Consider the nonlinear eq: $\langle \mathcal{R}(u), \varphi \rangle = (A(x, u) \nabla u, \nabla \varphi) - (f, \varphi) = 0.$

**Linearization iterations**

We generally solve nonlinear equations by linearization iterations, i.e., by finding a sequence $\{u^i_h\}_{i \in \mathbb{N}} \subset V_h \subset H^1_0(\Omega)$.

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Then defining the energy norms generated at iteration $i$ as

$$\|\varphi\|_{1,u^i_h} := \|A(x, u^i_h)\frac{1}{2} \nabla \varphi\| \quad \text{for } \varphi \in H^1_0(\Omega),$$

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we have robust estimates of $\|\mathcal{R}_{\text{lin}}^{u^i_h}(u^{i+1}_h)\|_{-1,u^i_h}$ (discretization error) but not of $\|\mathcal{R}(u^{i+1}_h)\|_{-1,u^i_h}$. 
1 Questions

- Can we get a robust estimate of $\| \mathcal{R}(u_h^i) \|_{-1,u_h^i}$ using the linearization iterations?

1 Questions

- Can we get a robust estimate of $\| \mathcal{R}(u_{ih}^i) \|_{-1, u_{ih}^i}$ using the linearization iterations?
- If yes, then can this be used to stop the iterations adaptively?

2 Outline

1 Introduction

2 Main analytical results
   Decomposition of error
   A posteriori error estimates

3 Scope of the results

4 Numerical results
2 An orthogonal decomposition result

Theorem 1  Decomposition of the total error using linearization

Under Assumption 1, and provided that the linearization iterations \( \{u_h^i\}_{i \in \mathbb{N}} \subset \mathcal{V}_h \subset H_0^1(\Omega) \) are generated by linear problems

\[
\langle R_{\text{lin}}^i(u), \varphi \rangle = \mathcal{L}(u_h^i; u - u_h^i, \varphi) + \langle R(u_h^i), \varphi \rangle = 0, \quad \forall \varphi \in H_0^1(\Omega),
\]

for a symmetric, bounded, coercive, bilinear form \( \mathcal{L}(u_h^i, \cdot, \cdot) \), and

\[
\|\varphi\|_{1,u_h^i} = \mathcal{L}(u_h^i; \varphi, \varphi)^{\frac{1}{2}}, \quad \|\zeta\|_{-1,u_h^i} = \sup_{\varphi \in H_0^1(\Omega)} \frac{\langle \zeta, \varphi \rangle}{\|\varphi\|_{1,u_h^i}},
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2 An orthogonal decomposition result

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\]

we have

\[
\|\| R(u_h^i) \|\|^2_{-1, u_h^i} = \|\| R_{\text{lin}}(u_h^{i+1}) \|\|^2_{-1, u_h^i} + \|\| u_h^{i+1} - u_h^i \|\|^2_{1, u_h^i}.
\]

\( \text{total error} \)

\( \text{discretization error of the linearization step} \)

\( \text{linearization error} \)
The linerization error is computed directly, we define

$$\eta_{\text{lin}, \Omega}^i := \| u_h^{i+1} - u_h^i \|_{1, u_h^i}^2.$$
2 A posteriori error estimates

The linerization error is computed directly, we define

\[ \eta_{\text{lin}, \Omega}^i := \left\| u_{h}^{i+1} - u_{h}^{i} \right\|_{1, u_{h}^i}^2. \]

For estimating \( \left\| R_{\text{lin}}^i u_{h}^{i+1} \right\|_{-1, u_{h}^i}^2 \), we introduce \( \eta_{\text{disc}, \Omega}^i \), following the analysis on robust estimates of singularly perturbed reaction-diffusion problems in [Smears & Vohralík (2020)], [Verfürth (1998), (2005)].
2 A posteriori error estimates

Theorem 2 Reliable, efficient, and robust a posteriori estimates

Global reliability

\[ \| \mathcal{R}(u^i_T) \|_{1, u^i_h}^2 \leq [\eta^i_\Omega]^2 = \sum_{K \in T} ([\eta^i_{\text{disc}, K}]^2 + [\eta^i_{\text{lin}, K}]^2). \]
2 A posteriori error estimates

Theorem 2 Reliable, efficient, and robust a posteriori estimates

Global reliability

\[ \| R(u_{iT}^i) \|_{1,u_h}^2 \leq [\eta^i_{\Omega}]^2 = \sum_{K \in T} ( [\eta^{i,\text{disc},K}]^2 + [\eta^{i,\text{lin},K}]^2 ). \]

Global efficiency

\[ [\eta^i_{\Omega}]^2 \lesssim \| R(u_{i,T}^i) \|_{1,u_h}^2 + \text{(data oscillation terms)}. \]
2 A posteriori error estimates

Theorem 2  Reliable, efficient, and robust a posteriori estimates

Global reliability

\[
\|R(u_i^T)\|_{-1,u_i^h}^2 \leq [\eta^i]_\Omega^2 = \sum_{K \in \mathcal{T}} ([\eta^i_{\text{disc},K}]^2 + [\eta^i_{\text{lin},K}]^2).
\]

Global efficiency

\[
[\eta^i_\Omega]^2 \lesssim \|R(u_i^h)\|_{-1,u_i^h}^2 + (\text{data oscillation terms}).
\]

Local efficiency

For \( \omega \subset \Omega \), there exists a neighbourhood \( \mathcal{U}_\omega \subseteq \Omega \) such that

\[
[\eta^i_\omega]^2 \lesssim \|R(u_{i+1}^h)\|_{-1,u_i^h,\mathcal{U}_\omega}^2 + [\eta^i_{\text{lin},\mathcal{U}_\omega}]^2 + (\text{data oscillation terms}).
\]
3 Outline

1. Introduction

2. Main analytical results

3. Scope of the results
   - Class of problems
   - Linearization schemes

4. Numerical results
3 Class of problems

Class 1: gradient independent diffusivity problems

For all \( \varphi \in H^1_0(\Omega) \), \( \mathcal{R} : H^1_0(\Omega) \rightarrow H^{-1}(\Omega) \) is defined as

\[
\langle \mathcal{R}(u), \varphi \rangle := \langle f(x, u), \varphi \rangle + \tau(\overline{K}(x)(D(x, u)\nabla u + q(x, u)), \nabla \varphi)
\]
3 Class of problems

Class 1: gradient independent diffusivity problems

For all $\varphi \in H^1_0(\Omega)$, $R : H^1_0(\Omega) \rightarrow H^{-1}(\Omega)$ is defined as

$$\langle R(u), \varphi \rangle := \langle f(x, u), \varphi \rangle + \tau(\bar{K}(x)(D(x, u)\nabla u + q(x, u))), \nabla \varphi \rangle$$

Semilinear equations $\Delta u = f(x, u)$

Such equations pop up in quantum mechanics (special solutions to nonlinear Klein-Gordon equations), gravitation influences on stars, membrane buckling problems etc.

Time-discrete nonlinear advection-reaction-diffusion equations

with time-step $\tau > 0$ following evolutions equations reduce to this case

Poro-Fischer equations: $\partial_t u = \Delta u^m + \lambda u (1 - u)$

Richards equation: $\partial_t S(u) = \nabla \cdot [\bar{K}(x)\kappa(S(u))(\nabla u + g)] + f(x, u)$

Biofilm equations: $\partial_t u_k = \mu_k \Delta \Phi_k(u_k) + f_k((u_k)^n_{k=1})$
3 Class of problems

Class 1: gradient independent diffusivity problems

For all $\varphi \in H^1_0(\Omega)$, $R : H^1_0(\Omega) \rightarrow H^{-1}(\Omega)$ is defined as

$$\langle R(u), \varphi \rangle := \langle f(x, u), \varphi \rangle + \tau(\bar{K}(x)(D(x, u)\nabla u + q(x, u)), \nabla \varphi)$$

Assumption 1 is satisfied if $\tau > 0$ is small, and

- $D : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^+$ is bounded and Lipschitz
- $\bar{K} : \Omega \rightarrow \mathbb{R}^{d \times d}$ is symmetric positive definite
- $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is monotone and Lipschitz up to the boundary
- $q : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^d$ is bounded, and satisfies a Lipschitz condition

with

$$\text{dist}(u, v) = \left\| \bar{K}^{\frac{1}{2}} \nabla \int_u^v D \right\|$$
3 Class of problems

Class 2: gradient dependent diffusivity problems

For all $\varphi \in H^1_0(\Omega)$, $\mathcal{R} : H^1_0(\Omega) \rightarrow H^{-1}(\Omega)$ is defined as

$$\langle \mathcal{R}(u), \varphi \rangle := \langle f(x, u), \varphi \rangle + (\sigma(x, \nabla u), \nabla \varphi)$$
3 Class of problems

Class 2: gradient dependent diffusivity problems

For all $\varphi \in H^1_0(\Omega)$, $\mathcal{R} : H^1_0(\Omega) \rightarrow H^{-1}(\Omega)$ is defined as

$$\langle \mathcal{R}(u), \varphi \rangle := \langle f(x, u), \varphi \rangle + (\sigma(x, \nabla u), \nabla \varphi)$$

For $a(\cdot)$ satisfying the ellipticity condition, and $b(\cdot) > 0$

- Mean curvature flow: $\sigma(x, y) = \left( a(x) + \frac{b(x)}{\sqrt{1+|y|^2}} \right) y$
- $p$-Laplacian problems*: $\sigma(x, y) = (a(x) + b(x)|y|^{p-2})y$
- Compressive flow*: $\sigma(x, y) = b(x) \left( 1 - \frac{r-1}{2} |y|^2 \right)^{\frac{1}{r-1}} y$
Class 2: gradient dependent diffusivity problems

For all $\varphi \in H^1_0(\Omega)$, $\mathcal{R} : H^1_0(\Omega) \to H^{-1}(\Omega)$ is defined as

$$\langle \mathcal{R}(u), \varphi \rangle := \langle f(x, u), \varphi \rangle + (\sigma(x, \nabla u), \nabla \varphi)$$

Assumption 1 is satisfied if $f(x, \cdot), \sigma(x, \cdot)$ is monotone and Lipschitz

$$(\sigma(x, y) - \sigma(x, z)) \cdot (y - z) \geq \lambda_m |y - z|^2 \quad \text{for } x \in \Omega \text{ and } y, z \in \mathbb{R}^d,$$

$$|\sigma(x, y) - \sigma(x, z)| \leq \lambda_M |y - z| \quad \text{for } x \in \Omega \text{ and } y, z \in \mathbb{R}^d.$$
Abstract linearization

For all $u_h^i \in V_h$, define the symmetric, coercive, and bounded bilinear form

$$\mathcal{L}(u_h^i; v, w) := (L(x, u_h^i) v, w) + (a(x, u_h^i) \nabla v, \nabla w).$$
3 Linearization schemes

Abstract linearization

For all \( u^i_h \in V_h \), define the symmetric, coercive, and bounded bilinear form

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Then we compute \( u^{i+1}_h \in V_h \) as the f.e. solution of the equation

\[
\mathcal{L}(u^i_h; u - u^i_h, \varphi) = -\langle \mathcal{R}(u^i_h), \varphi \rangle, \quad \forall \varphi \in H^1_0(\Omega).
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3 Linearization schemes

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For all $u_h^i \in V_h$, define the symmetric, coercive, and bounded bilinear form

$$\mathcal{L}(u_h^i; \nu, w) := (L(x, u_h^i) \nu, w) + (a(x, u_h^i) \nabla \nu, \nabla w).$$

Then we compute $u_h^{i+1} \in V_h$ as the f.e. solution of the equation

$$\mathcal{L}(u_h^i; u - u_h^i, \varphi) = -\langle R(u_h^i), \varphi \rangle, \quad \forall \varphi \in H_0^1(\Omega).$$

With respect to $\mathcal{L}$, the linearized energy norms are defined as

$$\|\varphi\|_{1,u_h^i} = \mathcal{L}(u_h^i; \varphi, \varphi)^{\frac{1}{2}} = \left( \int_{\Omega} L(x, u_h^i) \varphi^2 + |a(x, u_h^i) \frac{1}{2} \nabla \varphi|^2 \right)^{\frac{1}{2}},$$

$$\|\varsigma\|_{-1,u_h^i} = \sup_{\varphi \in H_0^1(\Omega)} \frac{\langle \varsigma, \varphi \rangle}{\|\varphi\|_{1,u_h^i}}.$$
3 Linearization schemes: practical examples

Abstract linearization

For \( \mathcal{L}(u_h^i; v, w) := (L(x, u_h^i) v, w) + (a(x, u_h^i) \nabla v, \nabla w) \), compute \( u_{h}^{i+1} \in V_h \) as the f.e. solution of the equation

\[
\mathcal{L}(u_h^i; u_h^{i+1} - u_h^i, \varphi) = -\langle \mathcal{R}(u_h^i), \varphi \rangle, \quad \forall \varphi \in H^1_0(\Omega).
\]

<table>
<thead>
<tr>
<th>Scheme</th>
<th>( L(x, v) )</th>
<th>( a(x, v)/\tau )</th>
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<tbody>
<tr>
<td>Picard</td>
<td>( \partial_\xi f(x, v) )</td>
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<td>( L ) (constant) ( \geq \frac{1}{2} \sup \partial_\xi f )</td>
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<td>( M )-scheme</td>
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Examples in gradient independent diffusivity case
3 Linearization schemes: practical examples

Abstract linearization

For $\mathcal{L}(u_h^i; v, w) := (L(x, u_h^i) v, w) + (a(x, u_h^i) \nabla v, \nabla w)$,
compute $u_h^{i+1} \in V_h$ as the f.e. solution of the equation

$\mathcal{L}(u_h^i; u_h^{i+1} - u_h^i, \varphi) = -\langle R(u_h^i), \varphi \rangle, \quad \forall \varphi \in H^1_0(\Omega)$.

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<tr>
<td>$L$-scheme</td>
<td>$L$ (constant) $\geq \frac{1}{2} \sup \partial_\xi f$</td>
<td>$\tilde{K}(x) D(x, v)$</td>
</tr>
<tr>
<td>$M$-scheme</td>
<td>$\partial_\xi f(x, v) + M\tau$ (constant)</td>
<td>$\tilde{K}(x) D(x, v)$</td>
</tr>
</tbody>
</table>

Examples in gradient independent diffusivity case

- Newton scheme leads to a non-symmetric $\mathcal{L}$ and is treated separately
3 Linearization schemes: practical examples

Abstract linearization

For $\mathcal{L}(u^i_h; v, w) := (L(x, u^i_h) v, w) + (a(x, u^i_h) \nabla v, \nabla w)$,
compute $u^{i+1}_h \in V_h$ as the f.e. solution of the equation

$\mathcal{L}(u^i_h; u^{i+1}_h - u^i_h, \varphi) = -\langle R(u^i_h), \varphi \rangle, \quad \forall \varphi \in H^1_0(\Omega)$.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$L(x, v)$</th>
<th>$a(x, v)/\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kačanov</td>
<td>$\partial_\xi f(x, v)$</td>
<td>$A(x,</td>
</tr>
<tr>
<td>Zarantonello</td>
<td>0</td>
<td>$\Lambda$ (constant) &gt; 0</td>
</tr>
</tbody>
</table>

Examples in gradient dependent diffusivity case
4 Outline

1 Introduction

2 Main analytical results

3 Scope of the results

4 Numerical results
   Gradient independent diffusivity
   Gradient independent diffusivity case
   The Newton scheme
4 Adaptive linearization & effectivity of estimates

Algorithm 1  Adaptive linearization

For a fixed $0 < \mu \ll 1$, we iterate until for some $i = \bar{i} \in \mathbb{N},$

$$\eta_{\text{lin},\Omega} \leq \mu [\eta_{\Omega}].$$
4 Adaptive linearization & effectivity of estimates

Algorithm 1 Adaptive linearization

For a fixed $0 < \mu \ll 1$, we iterate until for some $i = \tilde{i} \in \mathbb{N},$

$$\eta^i_{\text{lin},\Omega} \leq \mu [\eta^i_{\Omega}].$$

Effectivity indices

Global effectivity index: Eff. Ind. := $\frac{\eta^i_{\Omega}}{\| \mathcal{R}(u^i_h) \|_{-1,u^i_h,\Omega}}$

Local effectivity index: (Eff. Ind.)$_K$ := $\frac{\eta^i_K}{\| \mathcal{R}(u^i_h) \|_{-1,u^i,h,K}}$, $\forall K \in \mathcal{T},$
4 Adaptive linearization & effectivity of estimates

Algorithm 1 Adaptive linearization

For a fixed $0 < \mu \ll 1$, we iterate until for some $i = \bar{i} \in \mathbb{N}$,

$$\eta_{\text{lin},\Omega}^{\bar{i}} \leq \mu \lfloor \eta_{\Omega}^{\bar{i}} \rfloor.$$

Effectivity indices

Global effectivity index: Eff. Ind. := $\eta_{\Omega}^{i} / \| R(u_h^{i}) \|_{-1,u_h^{i}}$

Local effectivity index: $(\text{Eff. Ind.})_K := \eta_{K}^{i} / \| R(u_h^{i}) \|_{-1,u_h^{i},K}$, $\forall K \in \mathcal{T}$,

Mesh

Three mesh-levels used: $h = \frac{0.1}{\ell}$ where $\ell \in \{1, 2, 4\}$
4 Gradient independent diffusivity case: Richards equation

For $\Omega = (0, 1) \times (0, 1)$ we study

$$\langle R(\tilde{u}), \varphi \rangle = (S(\tilde{u}) - S(\bar{u}), \varphi)$$
$$+ \tau(\bar{K}_\kappa(S(\bar{u}))[\nabla \bar{u} - g], \nabla \varphi)$$

where the van Genuchten parametrization for $S, \kappa$ is used:

$$S(\xi) := \left(1 + (2 - \xi)\frac{1}{1-\lambda}\right)^{-\lambda},$$
$$\kappa(s) := \sqrt{s} \left(1 - (1 - s\frac{1}{\lambda})^\lambda\right)^2,$$

with $\lambda = 0.5$, $u_0^h = 0$,

$$\bar{K} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix}, \text{ and } g = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
4 Global effectivity

- Picard $\tau = 1.0$
- M-Scheme $\tau = 1.0$
- L-Scheme $\tau = 1.0$

- Picard $\tau = 0.01$
- M-Scheme $\tau = 0.01$
- L-Scheme $\tau = 0.01$
4 Distribution of error vs. estimates

Error $\mathbf{MS} \ l=2, \tau=1, \ i=9$

Error $\mathbf{MS} \ l=2, \tau=0.01, \ i=5$

Estimate

$n_k^i \ \mathbf{MS} \ l=2, \tau=1, \ i=9$

$n_k^i \ \mathbf{MS} \ l=2, \tau=0.01, \ i=5$
4 Local effectivity
For the adaptive stopping criteria $\mu = 0.05$ is chosen.
4 Gradient independent diffusivity case

We consider in $\Omega$ the equation

$$\varepsilon u - \nabla \cdot [A(|\nabla u|)\nabla u] = f$$

where

$$A(y) = 2 + \frac{y}{(1 + |y|^2)},$$

$\varepsilon = 10^{-2}$, and a singular $f \in H^{-1}(\Omega)$ is chosen such that the solution becomes

$$u_{\text{exact}} = r^4 \cos \left(\frac{4}{7} \theta\right).$$
4 Global effectivity and distribution of error
4 Local effectivity
4 Error with linearization iterations
4 The Newton scheme

For the Newton scheme, the linearization operator

\[ \mathcal{L}(u^i_h; \nu, w) := (L(x, u^i_h) \nu, w) + (a(x, u^i_h) \nabla \nu, \nabla w) + (w(x, u^i_h) \nu, \nabla w), \]

is non-symmetric.
4 The Newton scheme

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is non-symmetric. However, if for some \( C_N \in [0, 2) \) we have

\[ w(x, u_T^i) a^{-1}(x, u_T) w(x, u_T^i) \leq C_N^2 L(x, u_T^i), \quad \forall x \in \Omega, \text{ and } i \in \mathbb{N}, \]
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then,

\[ C_m(C_N) \left[ \left\| R_{\text{lin}}^{u_h^i}(u_T^{i+1}) \right\|_{-1, u_h^i}^2 + \left\| u_T^{i+1} - u_h^i \right\|_{1, u_h^i}^2 \right] \leq \left\| R(u_h^i) \right\|_{-1, u_h^i}^2 \]

\[ \leq C_M(C_N) \left[ \left\| R_{\text{lin}}^{u_h^i}(u_T^{i+1}) \right\|_{-1, u_h^i}^2 + \left\| u_T^{i+1} - u_h^i \right\|_{1, u_h^i}^2 \right] \]

with \( C_m(C_N), C_M(C_N) \to 1 \) if \( C_N \downarrow 0. \)
For gradient independent diffusivity case, we have

- **Global Effectivity**
- **Local Effectivity**
- **Error with iterations**
Thank you for your time