A multipoint stress-flux mixed finite element method for the Stokes-Biot fluid poroelastic structure interaction model

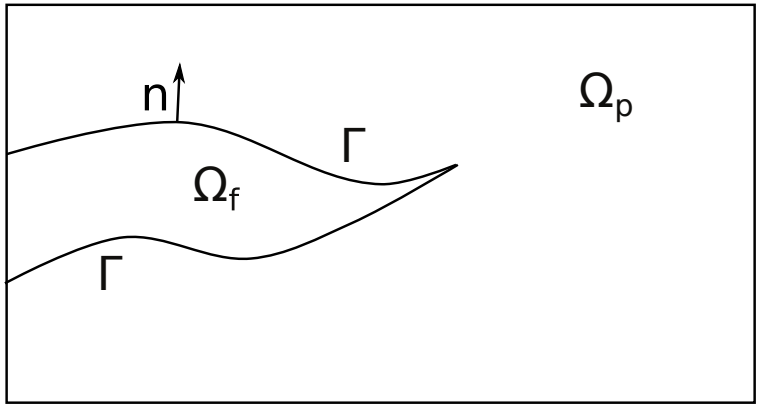
Ivan Yotov

Department of Mathematics
University of Pittsburgh

Sergio Caucao (Catolica University Cocepcion) and Tongtong Li (Dartmouth College)

InterPore 14th Annual Meeting, Abu Dhabi, May 30 - June 2, 2022

Stokes-Biot model for coupled flow with poroelastic structure



Biot system of poroelasticity in Ω_p:

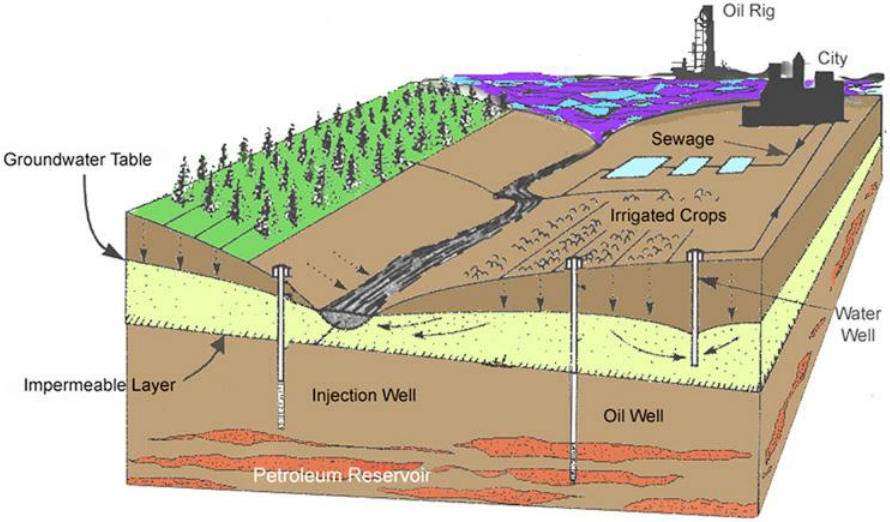
$$-\mathrm{div}\,\boldsymbol{\sigma}_p(\boldsymbol{\eta}_p) = \mathbf{f}_p, \quad \boldsymbol{\sigma}_p(\boldsymbol{\eta}_p) = \lambda_p(\mathrm{div}\,\boldsymbol{\eta}_p)\mathbf{I} + 2\mu_p\mathbf{D}(\boldsymbol{\eta}_p) - \alpha p_p\mathbf{I}$$

$$\frac{\partial}{\partial t}(s_0 p_p + \alpha\,\mathrm{div}\,\boldsymbol{\eta}_p) + \mathrm{div}\,\mathbf{u}_p = s, \quad K^{-1}\mathbf{u}_p = -\nabla p_p$$

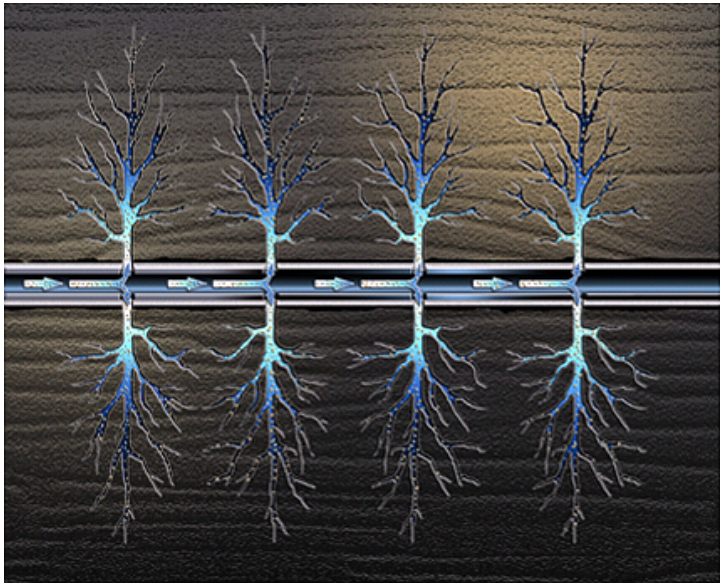
Stokes in Ω_f:

$$-\mathrm{div}\,\boldsymbol{\sigma}_f = \mathbf{f}_f, \quad \boldsymbol{\sigma}_f = -p_f\mathbf{I} + 2\mu_f\mathbf{D}(\mathbf{u}_f), \quad \mathrm{div}\,\mathbf{u}_f = g$$

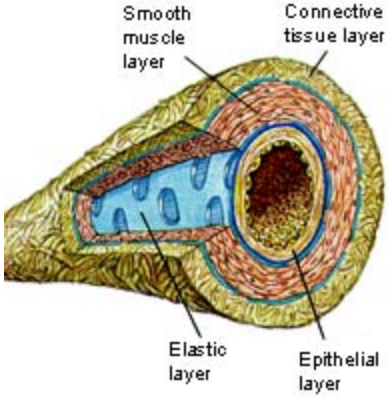
Applications of coupled flow and mechanics



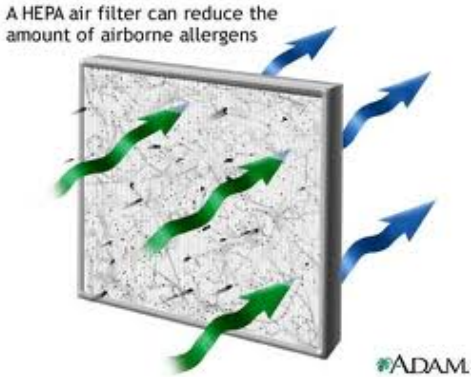
Surface-ground water systems



Hydraulic fracturing



Arterial flows



Industrial filters

Fully dual-mixed formulation and MFE discretization

- Darcy: velocity–pressure; multipoint flux mixed finite element method
- Elasticity: weakly-symmetric stress–displacement–rotation; multipoint stress mixed finite element method
- Stokes: weakly-symmetric stress–velocity–vorticity; multipoint stress mixed finite element method

Advantages:

- local mass conservation for the Darcy fluid; contiunous normal flux
- local momentum conservation for the solid and the Stokes fluid; continuous normal stress
- locking-free for almost incompressible material, small permeability K, and small storativity s_0
- local elimination of flux, stresses, rotation, vorticity: positive definite cell-centered pressure–displacement–velocity system

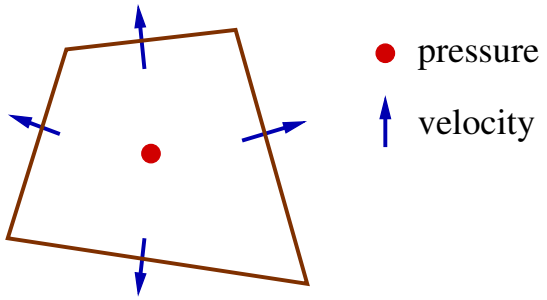
Discretization for Darcy flow

Model Problem:

$$\begin{aligned} \operatorname{div} \mathbf{u} &= f, && \text{in } \Omega \\ \mathbf{u} &= -K\nabla p, && \text{in } \Omega \end{aligned}$$

Mixed Finite Element (MFE):

$\mathbf{u}_h \in \mathbf{V}_h \subset H(\operatorname{div}; \Omega)$, $p_h \in W_h \subset L^2(\Omega)$:



$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v}) - (p_h, \operatorname{div} \mathbf{v}) &= 0, \qquad \forall \mathbf{v} \in \mathbf{V}_h \\ (\operatorname{div} \mathbf{u}_h, w) &= (f, w), \qquad \forall w \in W_h \end{aligned}$$

Multipoint Flux MFE - Accurate Cell-Centered Scheme [1]

Wheeler, Y. [2006], Klausen, Winther [2006]

Find $\mathbf{u}_h \in \mathbf{V}_h \subset H(\mathrm{div}\,;\Omega)$ and $p_h \in W_h \subset L^2(\Omega)$,

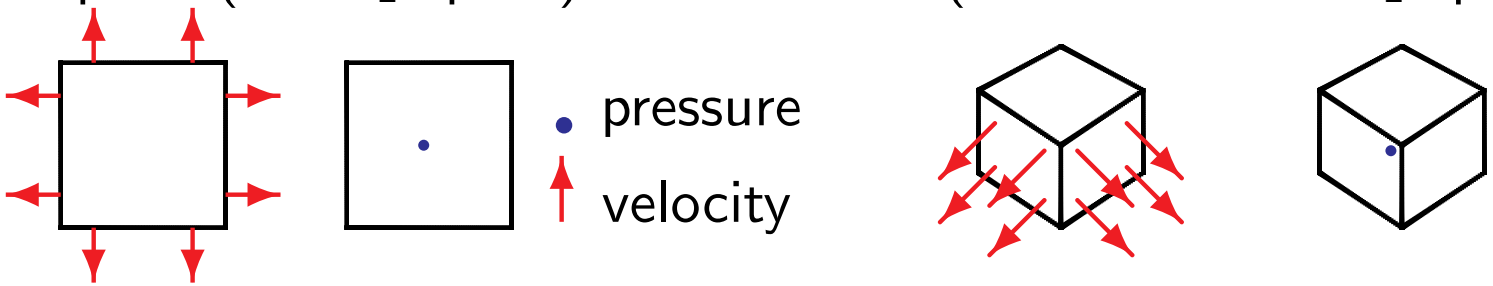
$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v})_Q - (p_h, \mathrm{div}\,\mathbf{v}) &= 0, \qquad \forall \mathbf{v} \in \mathbf{V}_h \\ (\mathrm{div}\,\mathbf{u}_h, w) &= (f, w), \qquad \forall w \in W_h \end{aligned}$$

1. Particular finite element spaces:
   The lowest order BDM_1 space
2. Specific numerical quadrature rule:
   Vertex rule for $(K^{-1}\mathbf{u}_h, \mathbf{v})_Q$

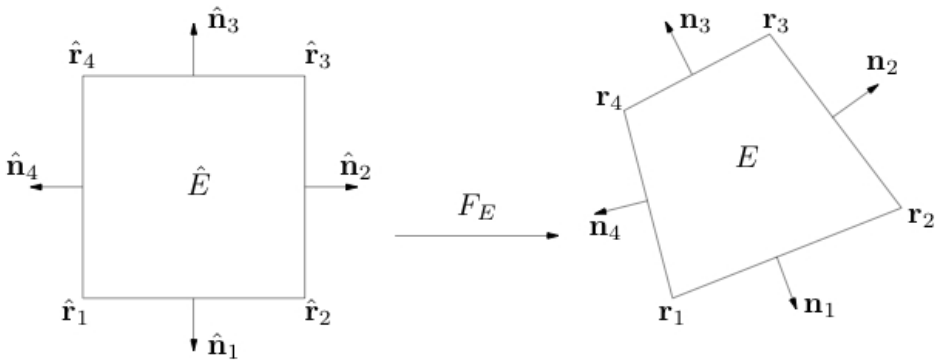
[1] Motivated by MPFA methods - Aavatsmark, Edwards, and collaborators

Mixed Finite Element Spaces

- 2D square (BDM_1 space) and 3D cube (enhanced $BDDF_1$ space):



Bilinear mapping



- DF_E: Jacobian
- $J_E := \det(DF_E)$

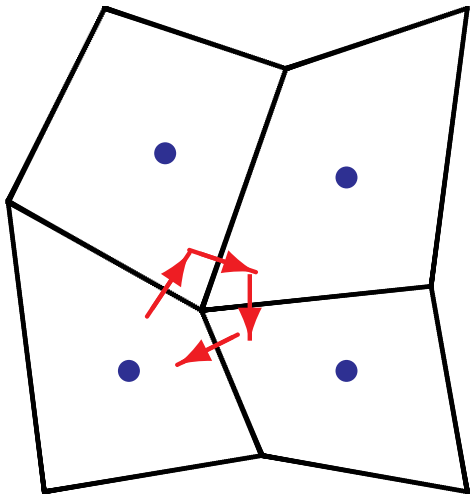
Piola transformation : $\mathbf{v} := \mathcal{P}\hat{\mathbf{v}} = \dfrac{1}{J_E} DF_E \hat{\mathbf{v}} \circ F_E^{-1}$

Reduction to a Cell-Centered Stencil

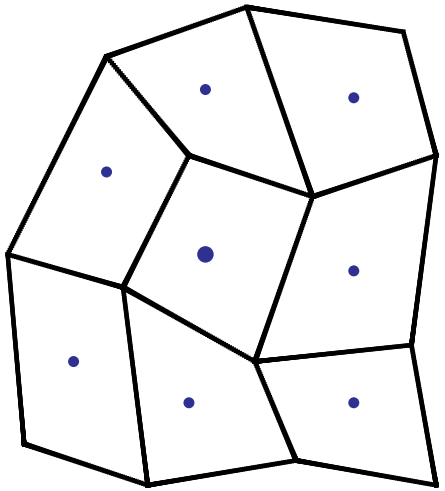
$$(K^{-1}\mathbf{u}_h, \mathbf{v}_h)_E = (\mathcal{M}_E \hat{\mathbf{u}}_h, \hat{\mathbf{v}}_h)_{\hat{E}}\,, \qquad \mathcal{M}_E = \frac{1}{J_E} DF_E^T K^{-1} DF_E$$

Numerical quadrature:

$$(K^{-1}\mathbf{u}_h, \mathbf{v}_h)_{Q,E} := (\mathcal{M}_E \hat{\mathbf{u}}_h, \hat{\mathbf{v}}_h)_{Q,\hat{E}} = \frac{1}{4}\sum_{i=1}^{4} \mathcal{M}_E(\hat{\mathbf{r}}_i)\hat{\mathbf{u}}_h(\hat{\mathbf{r}}_i)\cdot\hat{\mathbf{v}}_h(\hat{\mathbf{r}}_i),$$



Local velocity interaction



Cell-centered pressure stencil

Accuracy of Multipoint Flux MFE methods

Theorem (Ingram-Wheeler-Y. 2010)

For the symmetric MFMFE on smooth quadrilaterals and hexahedra

$$\|\mathbf{u} - \mathbf{u}_h\| + \|p - p_h\| \leq Ch(|\mathbf{u}|_1 + \|p\|_2) \equiv \mathcal{O}(h)$$

Theorem (Wheeler-Xue-Y. 2011)

For the non-symmetric MFMFE on general quadrilaterals and hexahedra

$$\|p - p_h\| + \|\Pi\mathbf{u} - \mathbf{u}_h\| \leq Ch(|\mathbf{u}|_1 + \|p\|_2) \equiv \mathcal{O}(h)$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{F}_h} \leq Ch(|\mathbf{u}|_1 + \|p\|_2) \equiv \mathcal{O}(h)$$

Face norm: $\|\mathbf{v}\|^2_{\mathcal{F}_h} := \sum_{E \in \mathcal{T}_h} \sum_{e \in \partial E} \frac{|E|}{|e|} \|\mathbf{v} \cdot \mathbf{n}_e\|^2_e \eqsim h \sum_{e \in \partial E} \|\mathbf{v} \cdot \mathbf{n}_e\|^2_e$

A multipoint stress MFE method for elasticity [2]

$$\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{\eta}) = \mathbf{f}, \quad \boldsymbol{\sigma}(\boldsymbol{\eta}) = \lambda(\operatorname{div}\boldsymbol{\eta})\mathbf{I} + 2\mu\mathbf{D}(\boldsymbol{\eta}), \quad \mathbf{D}(\boldsymbol{\eta}) = (\nabla\boldsymbol{\eta} + \nabla\boldsymbol{\eta}^T)/2$$

Stress–strain using compliance tensor:

$$\mathbf{A}\boldsymbol{\sigma}(\boldsymbol{\eta}) = \mathbf{D}(\boldsymbol{\eta}), \quad \mathbf{A}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{2\mu + d\lambda}\operatorname{tr}(\boldsymbol{\sigma})\mathbf{I}\right)$$

Weakly-symmetric mixed formulation (Arnold, Falk, Winther [2007]):

$$\mathbf{D}(\boldsymbol{\eta}) = \nabla\boldsymbol{\eta} - \boldsymbol{\gamma}, \quad \boldsymbol{\gamma} = \frac{1}{2}(\nabla\boldsymbol{\eta} - \nabla\boldsymbol{\eta}^T); \quad \mathbb{M} = \mathbb{R}^{d\times d}, \quad \mathbb{K} = \mathbb{R}^{d\times d}_{\text{skew}}$$

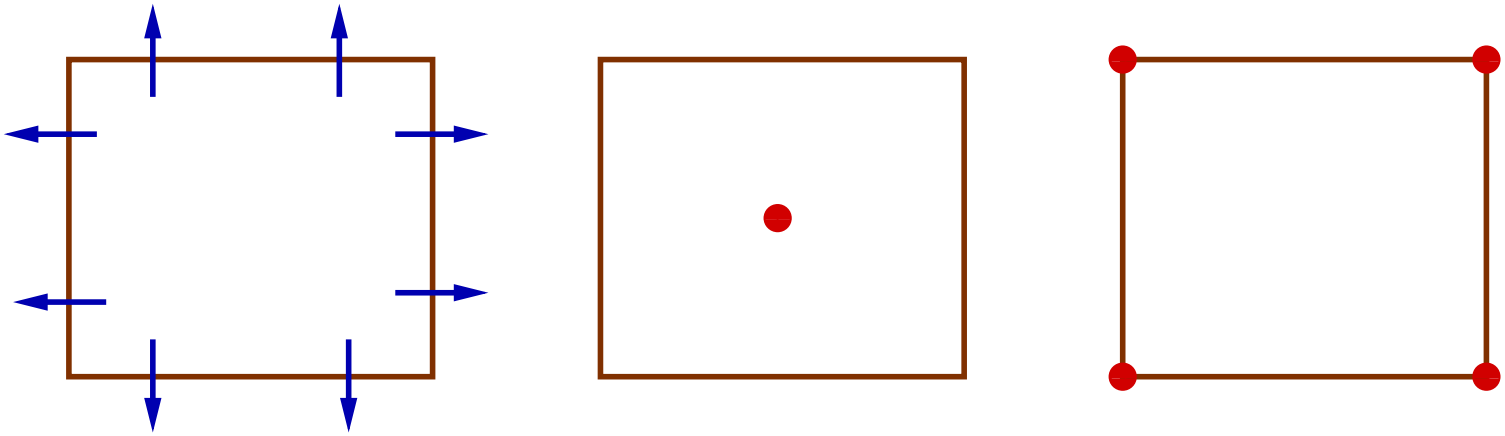
Find $(\boldsymbol{\sigma}, \boldsymbol{\eta}, \boldsymbol{\gamma}) \in H(\operatorname{div}, \Omega; \mathbb{M}) \times L^2(\Omega, \mathbb{V}) \times L^2(\Omega, \mathbb{K})$ such that

$$\begin{aligned}
&(\mathbf{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\tau}, \boldsymbol{\eta}) + (\boldsymbol{\tau}, \boldsymbol{\gamma}) = 0, && \boldsymbol{\tau} \in H(\operatorname{div}, \Omega; \mathbb{M}) \\
&(\operatorname{div}\boldsymbol{\sigma}, \boldsymbol{\xi}) = (\mathbf{f}, \boldsymbol{\xi}) && \boldsymbol{\xi} \in L^2(\Omega, \mathbb{V}) \\
&(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0, && \boldsymbol{\chi} \in L^2(\Omega, \mathbb{K})
\end{aligned}$$

[2] Ambartsumyan, Khattatov, Nordbotten, Y., SINUM 2020, Num Meth PDEs 2021

Multipoint stress MFE method

Mixed finite element spaces:



$\Sigma_h = (\mathrm{BDM}_1)^d$ $\quad$ $V_h = (P_0)^d$ $\quad$ $Q_h = (Q_1)^{d\times d}$

Find $(\boldsymbol{\sigma}_h, \boldsymbol{\eta}_h, \gamma_h) \in \Sigma_h \times V_h \times Q_h$ such that

$$
\begin{aligned}
&(\mathbf{A}\boldsymbol{\sigma}_h, \boldsymbol{\tau})_Q + (\operatorname{div}\boldsymbol{\tau}, \boldsymbol{\eta}_h) + (\boldsymbol{\tau}, \gamma_h)_Q = 0, && \boldsymbol{\tau} \in \Sigma_h \\
&(\operatorname{div}\boldsymbol{\sigma}_h, \boldsymbol{\xi}) = (\mathbf{f}, \boldsymbol{\xi}), && \boldsymbol{\xi} \in V_h \\
&(\boldsymbol{\sigma}_h, \chi)_Q = 0, && \chi \in Q_h
\end{aligned}
$$

Local stress and rotation elimination: cell-centered spd system for the displacement

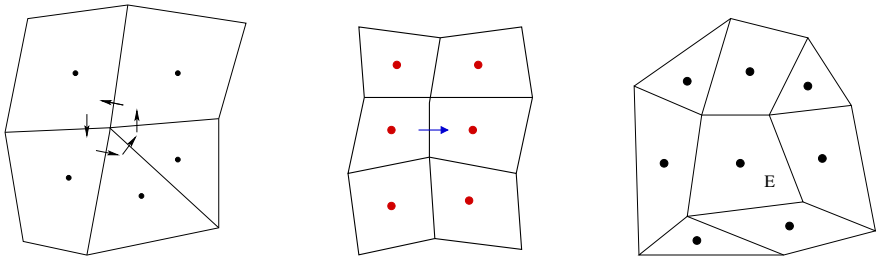
$$\begin{pmatrix} A_{\boldsymbol{\sigma}\boldsymbol{\sigma}} & A_{\boldsymbol{\sigma}\boldsymbol{\eta}}^T & A_{\boldsymbol{\sigma}\boldsymbol{\gamma}}^T \\ A_{\boldsymbol{\sigma}\boldsymbol{\eta}} & 0 & 0 \\ A_{\boldsymbol{\sigma}\boldsymbol{\gamma}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\eta} \\ \boldsymbol{\gamma} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \\ 0 \end{pmatrix}, \ \boldsymbol{\sigma} = -A_{\boldsymbol{\sigma}\boldsymbol{\sigma}}^{-1}(A_{\boldsymbol{\sigma}\boldsymbol{\eta}}^T\boldsymbol{\eta} + A_{\boldsymbol{\sigma}\boldsymbol{\gamma}}^T\boldsymbol{\gamma}),$$

where $A_{\boldsymbol{\sigma}\boldsymbol{\sigma}}$ is block-diagonal with blocks associated with vertices

$$-\begin{pmatrix} A_{\boldsymbol{\sigma}\boldsymbol{\eta}}A_{\boldsymbol{\sigma}\boldsymbol{\sigma}}^{-1}A_{\boldsymbol{\sigma}\boldsymbol{\eta}}^T & A_{\boldsymbol{\sigma}\boldsymbol{\eta}}A_{\boldsymbol{\sigma}\boldsymbol{\sigma}}^{-1}A_{\boldsymbol{\sigma}\boldsymbol{\gamma}}^T \\ A_{\boldsymbol{\sigma}\boldsymbol{\gamma}}A_{\boldsymbol{\sigma}\boldsymbol{\sigma}}^{-1}A_{\boldsymbol{\sigma}\boldsymbol{\eta}}^T & A_{\boldsymbol{\sigma}\boldsymbol{\gamma}}A_{\boldsymbol{\sigma}\boldsymbol{\sigma}}^{-1}A_{\boldsymbol{\sigma}\boldsymbol{\gamma}}^T \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta} \\ \boldsymbol{\gamma} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}$$

$A_{\boldsymbol{\sigma}\boldsymbol{\gamma}}A_{\boldsymbol{\sigma}\boldsymbol{\sigma}}^{-1}A_{\boldsymbol{\sigma}\boldsymbol{\gamma}}^T$ is block-diagonal with blocks associated with vertices

$$\boldsymbol{\gamma} = -(A_{\boldsymbol{\sigma}\boldsymbol{\gamma}}A_{\boldsymbol{\sigma}\boldsymbol{\sigma}}^{-1}A_{\boldsymbol{\sigma}\boldsymbol{\gamma}}^T)^{-1}A_{\boldsymbol{\sigma}\boldsymbol{\gamma}}A_{\boldsymbol{\sigma}\boldsymbol{\sigma}}^{-1}A_{\boldsymbol{\sigma}\boldsymbol{\eta}}^T\boldsymbol{\eta}$$



Stability of the MSMFE method

Babuska-Brezzi stability conditions:

$$(S1) \quad c\|\boldsymbol{\tau}\|_{H(div)} \leq (\mathbf{A}\boldsymbol{\tau}, \boldsymbol{\tau})_Q$$

$$\text{for } \boldsymbol{\tau} \in \Sigma_h : (\operatorname{div} \boldsymbol{\tau}, \boldsymbol{\xi}) + (\boldsymbol{\tau}, \mathbf{t})_Q = 0 \quad \forall (\boldsymbol{\xi}, \mathbf{t}) \in V_h \times Q_h$$

$$(S2) \qquad \inf_{(\boldsymbol{\xi},\boldsymbol{\chi}) \in V_h \times Q_h} \sup_{\boldsymbol{\tau} \in \Sigma_h} \frac{(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{\xi}) + (\boldsymbol{\tau}, \boldsymbol{\chi})_Q}{\|\boldsymbol{\tau}\|_{H(div)}(\|\boldsymbol{\xi}\|_{L^2} + \|\mathbf{t}\|_{L^2})} \geq \beta$$

Error estimate

Theorem (Ambartsumyan, Khattatov, Nordbotten, Y., SINUM 2020, Num Meth PDEs 2021)

On simplices and h^2-parallelograms,

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)} + \|div(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L^2(\Omega)} + \|\boldsymbol{\eta} - \boldsymbol{\eta}_h\|_{L^2(\Omega)} + \|\gamma - \gamma_h\|_{L^2(\Omega)} \leq Ch$$

$$\|P_h\boldsymbol{\eta} - \boldsymbol{\eta}_h\|_{L^2(\Omega)} \leq Ch^2$$

Stress-velocity-vorticity formulation for Stokes [3]

$$-\mathrm{div}\,\boldsymbol{\sigma}_f = \mathbf{f}_f, \quad \boldsymbol{\sigma}_f = -p_f\mathbf{I} + 2\mu_f\mathbf{D}(\mathbf{u}_f), \quad \mathrm{div}\,\mathbf{u}_f = 0$$

$$\mathrm{tr}(\boldsymbol{\sigma}_f) = \mathrm{tr}(-p_f\mathbf{I}) + 2\mu_f\,\mathrm{tr}(\mathbf{D}(\mathbf{u}_f)) = -dp_f + 2\mu_f\,\mathrm{div}\,\mathbf{u}_f = -dp_f$$

Deviatoric stress:

$$\boldsymbol{\sigma}_f^{\mathrm{d}} := \boldsymbol{\sigma}_f - \frac{1}{d}\mathrm{tr}(\boldsymbol{\sigma}_f)\mathbf{I}$$

Eliminate the pressure:

$$\frac{1}{2\mu_f}\boldsymbol{\sigma}_f^{\mathrm{d}} = \mathbf{D}(\mathbf{u}_f)$$

Weak symmetry: $\mathbf{D}(\mathbf{u}_f) = \nabla\mathbf{u}_f - \gamma_f$, $\gamma_f = \frac{1}{2}(\nabla\mathbf{u}_f - \nabla\mathbf{u}_f^T)$

Stress-velocity-vorticity formulation:

$$\frac{1}{2\mu_f}\boldsymbol{\sigma}_f^{\mathrm{d}} = \nabla\mathbf{u}_f - \gamma_f, \quad -\mathrm{div}\,\boldsymbol{\sigma}_f = \mathbf{f}_f$$

Recover the pressure from

$$p_f = -\frac{1}{d}\mathrm{tr}\boldsymbol{\sigma}_f$$

[3] Camano, Gatica, Oyarzua, Ruiz-Baier, Venegas, CMAME 2015

Multipoint stress mixed finite element method for Stokes

Mixed finite element spaces:

$$\Sigma_h \times V_h \times Q_h \subset H(\mathrm{div}, \Omega; \mathbb{M}) \times L^2(\Omega, \mathbb{V}) \times L^2(\Omega, \mathbb{K})$$

$$\Sigma_h \times V_h \times Q_h = (\mathrm{BDM}_1)^d \times (P_0)^d \times (Q_1)^{d \times d}$$

Find $(\boldsymbol{\sigma}_{f,h}, \mathbf{u}_{f,h}, \gamma_{f,h}) \in \Sigma_h \times V_h \times Q_h$ such that

$$\begin{aligned}
&\frac{1}{2\mu_{f,h}}(\boldsymbol{\sigma}^{\mathrm{d}}_{f,h}, \boldsymbol{\tau}^{\mathrm{d}}_{f,h})_Q + (\mathrm{div}\, \boldsymbol{\tau}_{f,h}, \mathbf{u}_{f,h}) + (\boldsymbol{\tau}_{f,h}, \gamma_{f,h})_Q = 0, && \boldsymbol{\tau}_{f,h} \in \Sigma_h \\
&(\mathrm{div}\, \boldsymbol{\sigma}_{f,h}, \boldsymbol{\xi}_{f,h}) = (\mathbf{f}_f, \boldsymbol{\xi}_{f,h}), && \boldsymbol{\xi}_{f,h} \in V_h \\
&(\boldsymbol{\sigma}_{f,h}, \chi_{f,h})_Q = 0, && \chi_{f,h} \in Q_h
\end{aligned}$$

Stability and convergence

$$\Sigma := H(\mathrm{div}, \Omega; \mathbb{M}) = \Sigma_0 + \mathbb{R}\mathbf{I}, \quad \Sigma_0 : \overline{\mathrm{tr}\boldsymbol{\tau}} = 0$$

It holds that[4]

$$C_{\mathrm{d}} \|\boldsymbol{\tau}_f\|^2 \leq \|\boldsymbol{\tau}_f^{\mathrm{d}}\|^2 + \|\nabla \cdot \boldsymbol{\tau}_f\|^2 \quad \forall \boldsymbol{\tau}_f \in \Sigma_0.$$

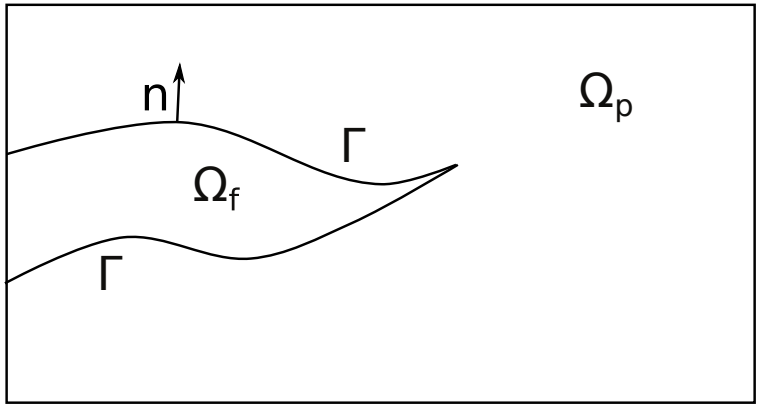
Theorem

There exists a unique solution to the MSMFE method for Stokes. On simplices and h^2-parallelograms,

$$\|\boldsymbol{\sigma}_f - \boldsymbol{\sigma}_{f,h}\|_{H(div;\Omega)} + \|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^2(\Omega)} + \|\boldsymbol{\gamma}_f - \boldsymbol{\gamma}_{f,h}\|_{L^2(\Omega)} \leq Ch$$

[4] Gatica, Springer 2014

Stokes-Biot model for coupled flow with poroelastic structure



Biot system of poroelasticity in Ω_p:

$$-\mathrm{div}\,\boldsymbol{\sigma}_p(\boldsymbol{\eta}_p) = \mathbf{f}_p, \quad \boldsymbol{\sigma}_p(\boldsymbol{\eta}_p) = \lambda_p(\mathrm{div}\,\boldsymbol{\eta}_p)\mathbf{I} + 2\mu_p\mathbf{D}(\boldsymbol{\eta}_p) - \alpha p_p\mathbf{I}$$

$$\frac{\partial}{\partial t}(s_0 p_p + \alpha\,\mathrm{div}\,\boldsymbol{\eta}_p) + \mathrm{div}\,\mathbf{u}_p = g_p, \quad K^{-1}\mathbf{u}_p = -\nabla p_p$$

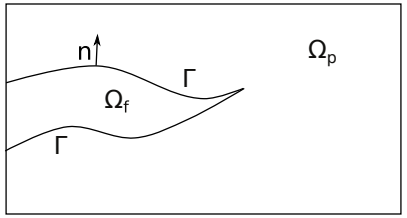
Stokes in Ω_f:

$$-\mathrm{div}\,\boldsymbol{\sigma}_f = \mathbf{f}_f, \quad \boldsymbol{\sigma}_f = -p_f\mathbf{I} + 2\mu_f\mathbf{D}(\mathbf{u}_f), \quad \mathrm{div}\,\mathbf{u}_f = 0$$

Biot-Stokes interface conditions on Γ

Showalter [2000]

Badia, Quaini, Quarteroni [2009]



Mass conservation:

$$\mathbf{u}_f \cdot \mathbf{n}_f + (\partial_t \boldsymbol{\eta}_p + \mathbf{u}_p) \cdot \mathbf{n}_p = 0$$

Beavers-Joseph-Saffman condition (slip with friction):

$$(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_f = -c_{BJS}(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p) \cdot \boldsymbol{\tau}_f$$

Balance of normal fluid stress:

$$(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = -p_p$$

Conservation of momentum:

$$\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = 0$$

Fully mixed Stokes-Biot formulation

$$
\begin{aligned}
\frac{1}{2\mu}\left(\boldsymbol{\sigma}_f^{\mathrm{d}}, \boldsymbol{\tau}_f^{\mathrm{d}}\right)_{\Omega_f} + \left(\mathbf{u}_f, \operatorname{div}\boldsymbol{\tau}_f\right)_{\Omega_f} - \left\langle \boldsymbol{\tau}_f \mathbf{n}_f, \boldsymbol{\varphi}\right\rangle_{\Gamma_{fp}} + \left(\boldsymbol{\gamma}_f, \boldsymbol{\tau}_f\right)_{\Omega_f} &= 0,\\
\mu\left(\mathbf{K}^{-1}\mathbf{u}_p, \mathbf{v}_p\right)_{\Omega_p} - \left(p_p, \nabla\cdot\mathbf{v}_p\right)_{\Omega_p} + \left\langle \mathbf{v}_p\cdot\mathbf{n}_p, \lambda\right\rangle_{\Gamma_{fp}} &= 0,\\
s_0\left(\partial_t p_p, q_p\right)_{\Omega_p} + \alpha_p\left(\partial_t A\boldsymbol{\sigma}_p, q_p\mathbf{I}\right)_{\Omega_p} + \alpha_p\left(\partial_t \alpha_p A p_p\mathbf{I}\right), q_p\mathbf{I})_{\Omega_p}&\\
+\left(q_p, \nabla\cdot\mathbf{u}_p\right)_{\Omega_p} &= \left(g_p, q_p\right)_{\Omega_p},\\
\left(\partial_t A\boldsymbol{\sigma}_p, \boldsymbol{\tau}_p\right)_{\Omega_p} + \alpha_p\left(\partial_t p_p, \operatorname{tr}(A\boldsymbol{\tau}_p)\right)_{\Omega_p} + \left(\mathbf{u}_s, \operatorname{div}\boldsymbol{\tau}_p\right)_{\Omega_p} - \left\langle \boldsymbol{\tau}_p\mathbf{n}_p, \boldsymbol{\theta}\right\rangle_{\Gamma_{fp}}&\\
+\left(\boldsymbol{\gamma}_p, \boldsymbol{\tau}_p\right)_{\Omega_p} &= 0,\\
-\left(\mathbf{v}_f, \operatorname{div}\boldsymbol{\sigma}_f\right)_{\Omega_f} &= \left(\mathbf{f}_f, \mathbf{v}_f\right)_{\Omega_f},\\
-\left(\mathbf{v}_s, \operatorname{div}\boldsymbol{\sigma}_p\right)_{\Omega_p} &= \left(\mathbf{f}_p, \mathbf{v}_s\right)_{\Omega_p},\\
-\left(\boldsymbol{\sigma}_f, \boldsymbol{\chi}_f\right)_{\Omega_f} &= 0,\\
-\left(\boldsymbol{\sigma}_p, \boldsymbol{\chi}_p\right)_{\Omega_p} &= 0,\\
-\left\langle \boldsymbol{\varphi}\cdot\mathbf{n}_f + \left(\boldsymbol{\theta}+\mathbf{u}_p\right)\cdot\mathbf{n}_p, \xi\right\rangle_{\Gamma_{fp}} &= 0,\\
\left\langle \boldsymbol{\phi}\cdot\mathbf{n}_p, \lambda\right\rangle_{\Gamma_{fp}} - c_{\mathrm{BJS}}\left\langle \sqrt{\mathbf{K}^{-1}}\left(\boldsymbol{\varphi}-\boldsymbol{\theta}\right)\cdot\boldsymbol{\tau}_f, \boldsymbol{\phi}\cdot\boldsymbol{\tau}_f\right\rangle_{\Gamma_{fp}} + \left\langle \boldsymbol{\sigma}_p\mathbf{n}_p, \boldsymbol{\phi}\right\rangle_{\Gamma_{fp}} &= 0,\\
\left\langle \boldsymbol{\sigma}_f\mathbf{n}_f, \boldsymbol{\psi}\right\rangle_{\Gamma_{fp}} + c_{\mathrm{BJS}}\left\langle \sqrt{\mathbf{K}^{-1}}\left(\boldsymbol{\varphi}-\boldsymbol{\theta}\right)\cdot\boldsymbol{\tau}_f, \boldsymbol{\psi}\cdot\boldsymbol{\tau}_f\right\rangle_{\Gamma_{fp}} + \left\langle \boldsymbol{\psi}\cdot\mathbf{n}_f, \lambda\right\rangle_{\Gamma_{fp}} &= 0.
\end{aligned}
$$

The MSMFE-MFMFE method for Stokes-Biot

- Simplicial or quadrilateral finite element partitions of Ω_f and Ω_p
- Allow for non-matching grids across Γ_{fp}
- Stokes: $(\mathrm{BDM}_1)^d \times (P_0)^d \times (P_1)^{d\times d}$ for $(\boldsymbol{\sigma}_f, \mathbf{u}_f, \gamma_f)$
- Elasticity: $(\mathrm{BDM}_1)^d \times (P_0)^d \times (P_1)^{d\times d}$ for $(\boldsymbol{\sigma}_p, \mathbf{u}_s, \gamma_p)$
- Darcy: $\mathrm{BDM}_1 \times P_0$ for $(\mathbf{u}_p, p_p)$
- Interface fluid velocity: $\Lambda_{f,h} = \Sigma_{f,h}\mathbf{n}_f|_{\Gamma_{fp}} = (P_1^{dc})^d$ for $\boldsymbol{\varphi}$
- Interface structure velocity: $\Lambda_{s,h} = \Sigma_{p,h}\mathbf{n}_p|_{\Gamma_{fp}} = (P_1^{dc})^d$ for $\boldsymbol{\theta}$
- Interface pressure: $\Lambda_{p,h} = \mathbf{V}_{p,h} \cdot \mathbf{n}_p|_{\Gamma_{fp}} = P_1^{dc}$ for λ

Theorem

The method is well-posed and first order accurate.

The full algebraic system

$$
\begin{pmatrix}
A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f} & 0 & 0 & 0 & A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\mathbf{u}_f} & 0 & A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} & 0 & A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0 & 0 \\
0 & A_{\mathbf{u}_p\mathbf{u}_p} & A^{\mathrm{t}}_{\mathbf{u}_p p_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{\mathrm{t}}_{\mathbf{u}_p\lambda} \\
0 & -A_{\mathbf{u}_p p_p} & A_{p_p p_p} & A^{\mathrm{t}}_{p_p\boldsymbol{\sigma}_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{p_p\boldsymbol{\sigma}_p} & A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p} & 0 & A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 & A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} & 0 & A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\boldsymbol{\theta}} & 0 \\
-A_{\boldsymbol{\sigma}_f\mathbf{u}_f} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -A_{\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{\boldsymbol{\varphi}\boldsymbol{\varphi}} & A^{\mathrm{t}}_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A^{\mathrm{t}}_{\boldsymbol{\varphi}\lambda} \\
0 & 0 & 0 & -A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}} & 0 & 0 & 0 & 0 & A_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A_{\boldsymbol{\theta}\boldsymbol{\theta}} & A^{\mathrm{t}}_{\boldsymbol{\theta}\lambda} \\
0 & -A_{\mathbf{u}_p\lambda} & 0 & 0 & 0 & 0 & 0 & 0 & -A_{\boldsymbol{\varphi}\lambda} & -A_{\boldsymbol{\theta}\lambda} & 0
\end{pmatrix}
$$

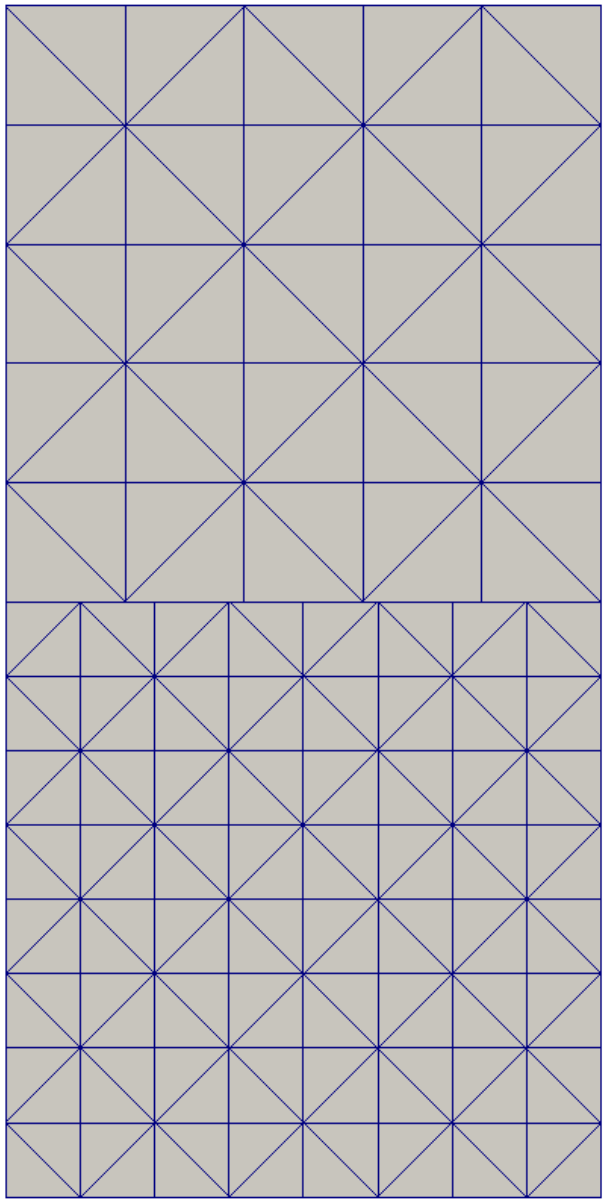
The reduced cell centered & Lagrange multiplier system

$$\begin{pmatrix} \widetilde{A}_{p_p\mathbf{u}_p p_p} & 0 & \widetilde{A}_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 & \widetilde{A}_{p_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A_{p_p\mathbf{u}_p\lambda} \\ 0 & \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} & 0 & \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0 & 0 \\ \widetilde{A}^{\mathrm{t}}_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 & \widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 & \widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} & 0 \\ 0 & \widetilde{A}^{\mathrm{t}}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0 & \widetilde{A}_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & A^{\mathrm{t}}_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A^{\mathrm{t}}_{\boldsymbol{\varphi}\lambda} \\ \widetilde{A}^{\mathrm{t}}_{p_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} & 0 & \widetilde{A}^{\mathrm{t}}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A_{\boldsymbol{\varphi}\boldsymbol{\theta}} & \widetilde{A}_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A^{\mathrm{t}}_{\boldsymbol{\theta}\lambda} \\ A^{\mathrm{t}}_{p_p\mathbf{u}_p\lambda} & 0 & 0 & -A_{\boldsymbol{\varphi}\lambda} & -A_{\boldsymbol{\theta}\lambda} & A_{\lambda\mathbf{u}_p\lambda} \end{pmatrix} \begin{pmatrix} p_p \\ \mathbf{u}_f \\ \mathbf{u}_s \\ \boldsymbol{\varphi} \\ \boldsymbol{\theta} \\ \lambda \end{pmatrix} = \begin{pmatrix} F_{p_p} \\ F_{\mathbf{u}_f} \\ F_{\mathbf{u}_s} \\ F_{\boldsymbol{\varphi}} \\ F_{\boldsymbol{\theta}} \\ F_{\lambda} \end{pmatrix}$$

Lemma

The reduced matrix is positive definite.

Convergence test



$$\mathbf{u}_f = \pi \cos(\pi t) \begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix}$$

$$p_f = \mathbf{e}^t \sin(\pi x) \cos(\frac{\pi y}{2}) + 2\pi \cos(\pi t)$$

$$\mathbf{u}_p = \pi \mathbf{e}^t \begin{pmatrix} \cos(\pi x) \cos(\frac{\pi y}{2}) \\ \frac{1}{2} \sin(\pi x) \sin(\frac{\pi y}{2}) \end{pmatrix}$$

$$p_p = \mathbf{e}^t \sin(\pi x) \cos(\frac{\pi y}{2})$$

$$\boldsymbol{\eta}_p = \sin(\pi t) \begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix}$$

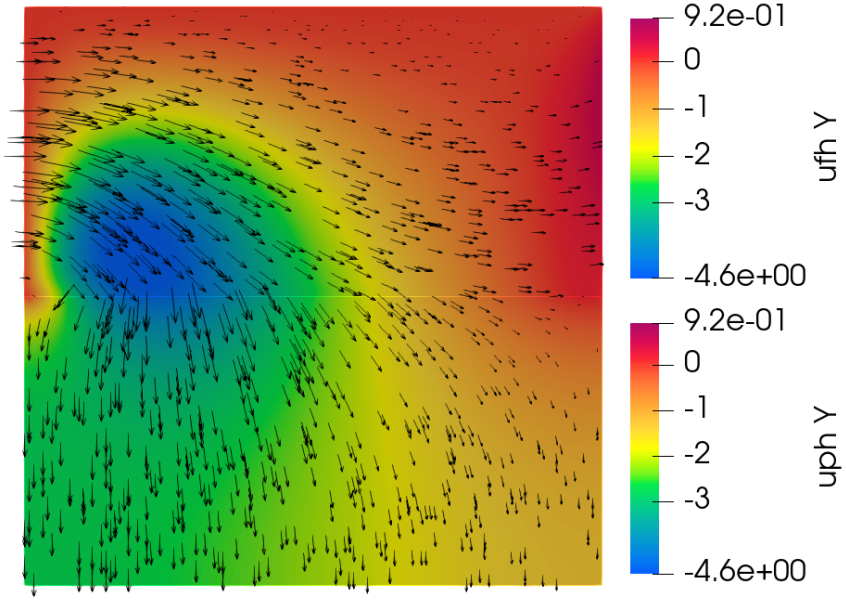
Numerical errors and convergence rates

h_f	$\|\epsilon(\boldsymbol{\sigma}_f)\|_{l^2(0,T;\mathbb{X}_f)}$ error	rate	$\|\epsilon(\mathbf{u}_f)\|_{l^2(0,T;\mathbf{V}_f)}$ error	rate	$\|\epsilon(\boldsymbol{\gamma}_f)\|_{l^2(0,T;\mathbb{Q}_f)}$ error	rate	$\|\epsilon(p_f)\|_{l^2(0,T;L^2(\Omega_f))}$ error	rate
0.7071	2.9664	–	3.7507	–	0.5247	–	0.8067	–
0.3727	1.5562	1.0072	1.9440	1.0261	0.1959	1.5382	0.4113	1.0519
0.1964	0.6987	1.2502	0.8646	1.2651	0.0715	1.5739	0.2001	1.1246
0.0997	0.3672	0.9485	0.4381	1.0022	0.0285	1.3566	0.0986	1.0443
0.0487	0.1806	0.9902	0.2149	0.9940	0.0130	1.0926	0.0508	0.9264
0.0250	0.0901	1.0406	0.1070	1.0435	0.0062	1.1112	0.0248	1.0730

h_p	$\|\epsilon(\boldsymbol{\sigma}_p)\|_{l^\infty(0,T;\mathbb{X}_p)}$ error	rate	$\|\epsilon(\mathbf{u}_s)\|_{l^2(0,T;\mathbf{V}_p)}$ error	rate	$\|\epsilon(\boldsymbol{\gamma}_p)\|_{l^2(0,T;\mathbb{Q}_p)}$ error	rate	$\|\epsilon(\mathbf{u}_p)\|_{l^2(0,T;\mathbf{X}_p)}$ error	rate	$\|\epsilon(p_p)\|_{l^\infty(0,T;\mathcal{Q}_p)}$ error	rate
0.4779	0.4546	–	2.3537	–	3.3381	–	5.3933	–	0.1331	–
0.2652	0.2228	1.2103	1.1492	1.2170	0.6761	2.7107	2.7667	1.1332	0.0664	1.1816
0.1267	0.1090	0.9679	0.5637	0.9647	0.1784	1.8045	1.2575	1.0679	0.0303	1.0611
0.0637	0.0561	0.9675	0.2845	0.9950	0.0448	2.0124	0.6293	1.0074	0.0155	0.9718
0.0349	0.0282	1.1435	0.1432	1.1418	0.0145	1.8708	0.3137	1.1579	0.0079	1.1338
0.0210	0.0141	1.3613	0.0713	1.3693	0.0049	2.1358	0.1548	1.3857	0.0039	1.3718

h_{tf}	$\|\epsilon(\boldsymbol{\varphi})\|_{l^2(0,T;\mathbf{L}^2(\Gamma_{fp}))}$ error	rate	h_{tp}	$\|\epsilon(\boldsymbol{\theta})\|_{l^2(0,T;\mathbf{L}^2(\Gamma_{fp}))}$ error	rate	$\|\epsilon(\lambda)\|_{l^2(0,T;L^2(\Gamma_{fp}))}$ error	rate
1/2	0.2540	–	1/3	0.4758	–	0.0990	–
1/4	0.0516	2.2998	1/6	0.2101	1.1792	0.0269	1.8800
1/8	0.0107	2.2673	1/12	0.0628	1.7427	0.0074	1.8599
1/16	0.0021	2.3308	1/24	0.0133	2.2347	0.0017	2.1024
1/32	0.0004	2.3059	1/48	0.0035	1.9267	0.0004	1.9644
1/64	0.0001	2.1602	1/96	0.0008	2.1900	0.0001	2.1079

Coupled surface and subsurface flows

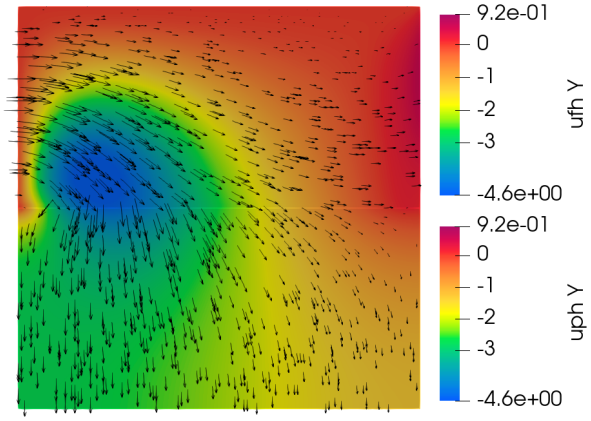


$$
\begin{aligned}
&\mathbf{u}_f = (-40y(y-1),\ \ 0)^t && \text{on} \quad \Gamma_{f,left},\\
&\mathbf{u}_f = 0 && \text{on} \quad \Gamma_{f,top},\\
&\boldsymbol{\sigma}_f \mathbf{n}_f = 0 && \text{on} \quad \Gamma_{f,right},\\
&p_p = 0 \quad \text{and} \quad \boldsymbol{\sigma}_p \mathbf{n}_p = 0 && \text{on} \quad \Gamma_{p,bottom},\\
&\mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{and} \quad \mathbf{u}_s = 0 && \text{on} \quad \Gamma_{p,left} \cup \Gamma_{p,right}.
\end{aligned}
$$

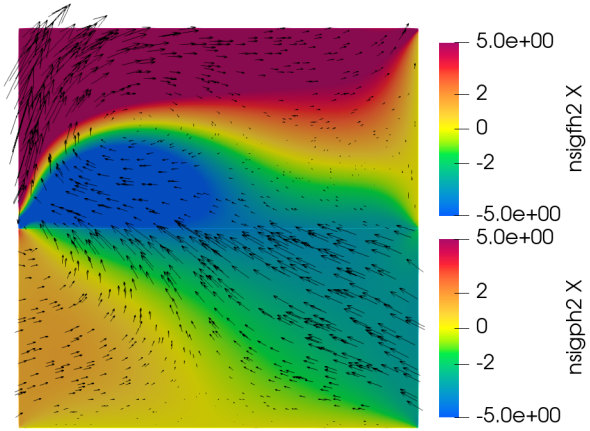
$$\mu = 1, \quad \alpha_p = 1, \quad \lambda_p = 1, \quad \mu_p = 1, \quad s_0 = 1, \quad \mathbf{K} = \mathbf{I}, \quad \alpha_{\mathrm{BJS}} = 1.$$

$$\mathbf{f}_f = 0, \quad q_f = 0, \quad \mathbf{f}_p = 0, \quad q_p = 0, \quad p_{p,0} = 0, \quad \text{and} \quad \boldsymbol{\sigma}_{p,0} = 0.$$

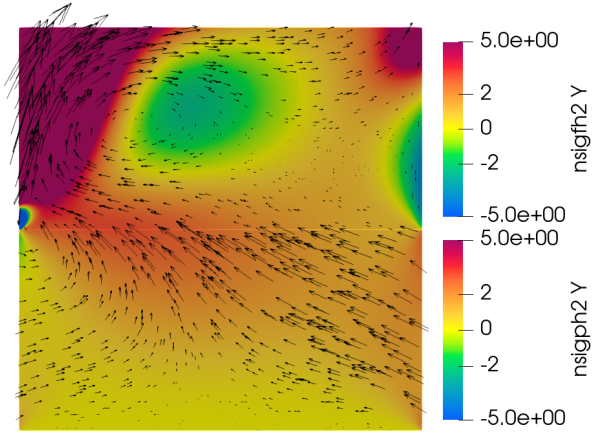
Coupled surface and subsurface flows



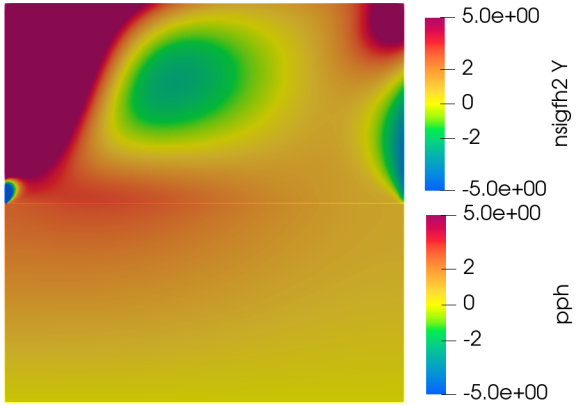
Velocity



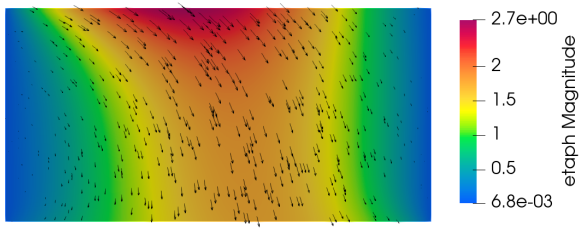
Vertical stress x



Vertical stress y

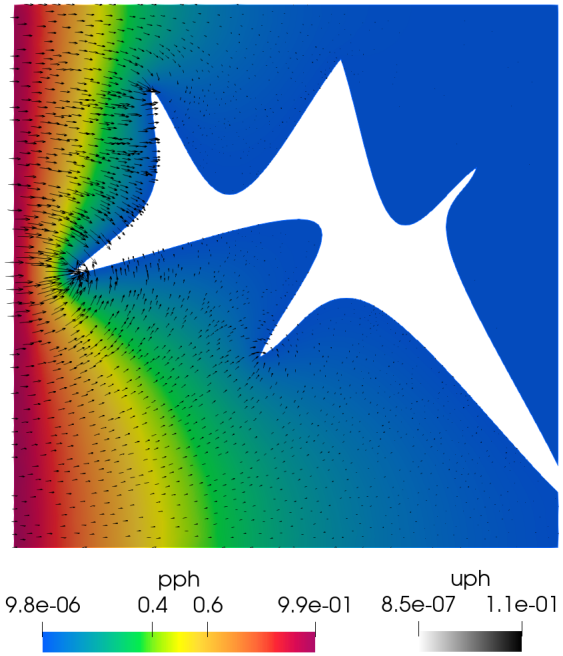


Stress - pressure



Displacement

Fluid filled cavity



$$\mu = 10^{-6} \text{ kPa s}, \quad \alpha_p = 1, \quad \lambda_p = 5/18 \times 10^7 \text{ kPa}, \quad \mu_p = 5/12 \times 10^7 \text{ kPa},$$

$$s_0 = 6.89 \times 10^{-2} \text{ kPa}^{-1}, \quad \mathbf{K} = 10^{-8} \times \mathbf{I} \text{ m}^2, \quad \alpha_{\mathrm{BJS}} = 1.$$

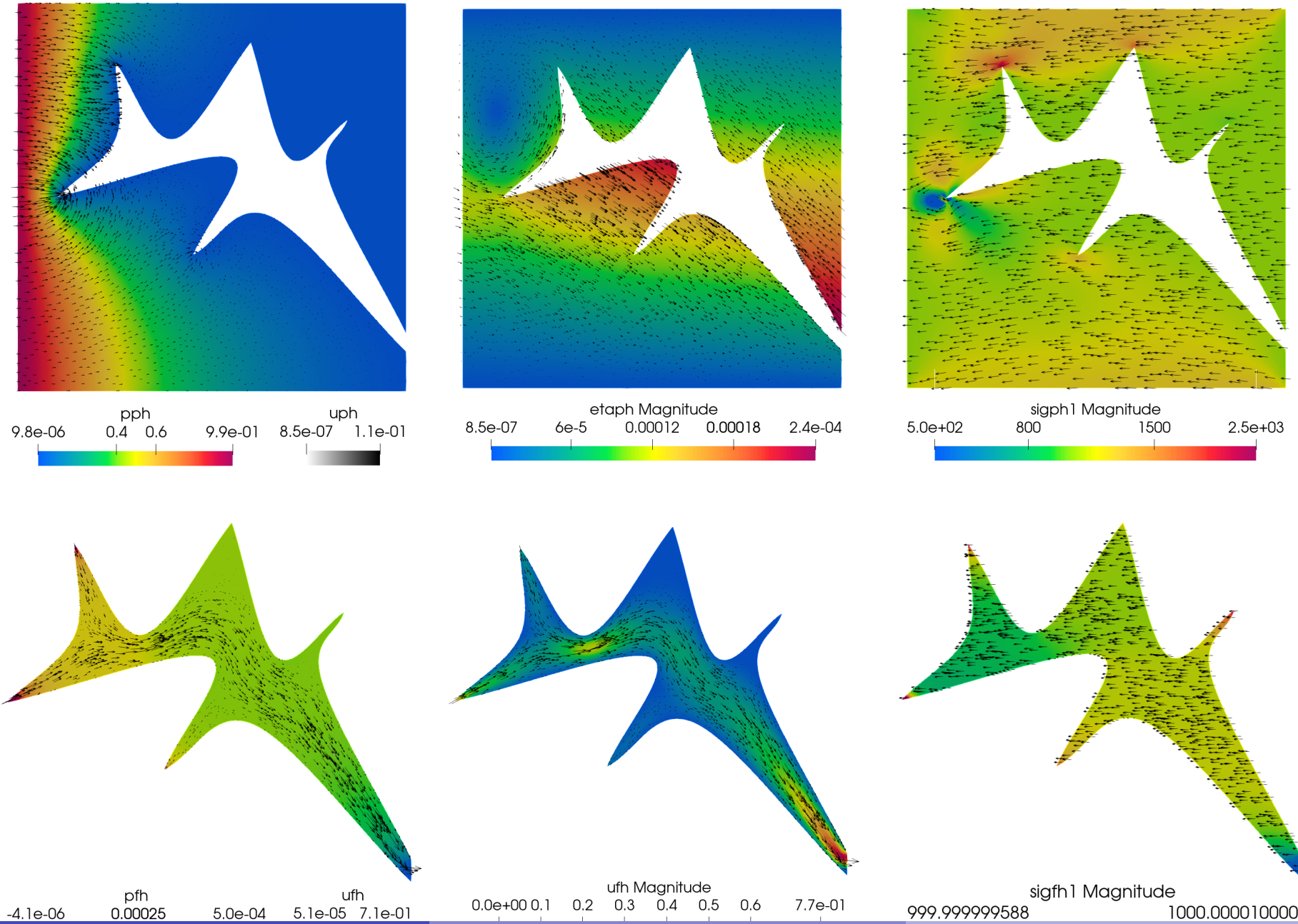
$$\mathbf{f}_f = 0, \quad q_f = 0, \quad \mathbf{f}_p = 0, \quad q_p = 0, \quad p_{p,0} = 1000, \quad \text{and} \quad \boldsymbol{\sigma}_{p,0} = -\alpha_p \, p_{p,0} \, \mathbf{I}.$$

$$\boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{n}_f = 1000, \quad \mathbf{u}_f \cdot \mathbf{t}_f = 0 \quad \text{on} \quad \Gamma_{f,right},$$

$$p_p = 1001 \quad \text{on} \quad \Gamma_{p,left}, \quad p_p = 1000 \quad \text{on} \quad \Gamma_{p,right} \quad \text{and} \quad \mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_{p,top} \cup \Gamma_{p,bottom},$$

$$\boldsymbol{\sigma}_p \, \mathbf{n}_p = -\alpha_p \, p_p \, \mathbf{n}_p \quad \text{on} \quad \Gamma_{p,left} \cup \Gamma_{p,right} \quad \text{and} \quad \mathbf{u}_s = 0 \quad \text{on} \quad \Gamma_{p,top} \cup \Gamma_{p,bottom}.$$

Fluid filled cavity



Extensions

- Navier–Stokes – Biot model
- Non-symmetric version for general quads
- Extension to hexahedra

Reference: Sergio Caucao, Tongtong Li, Ivan Yotov, *A multipoint stress-flux mixed finite element method for the Stokes-Biot model*, arxiv.org/abs/2011.01396 [math.NA]

Funding: NSF grant DMS 1818775; ANID-Chile project PAI77190084