

Numerical Analysis of a Mixed Finite Element Approximation of a Model of Biofilm Growth in Porous Media

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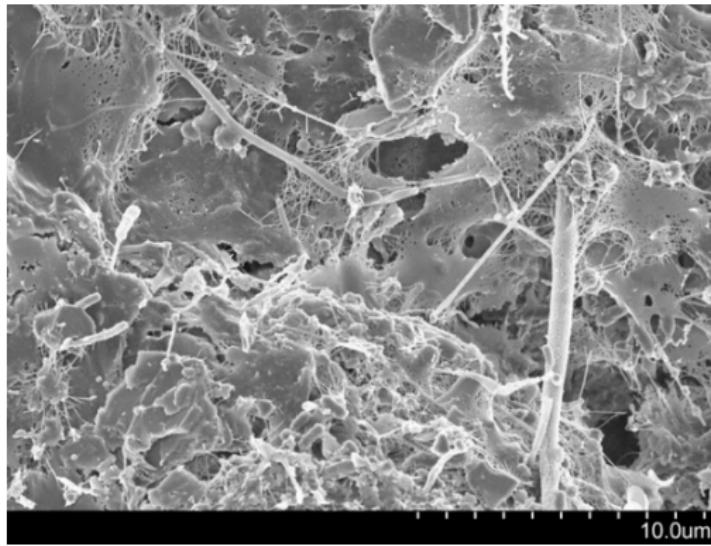
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The 14th Annual Meeting

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Biofilm Growth in Porous Media

Biofilm Structure

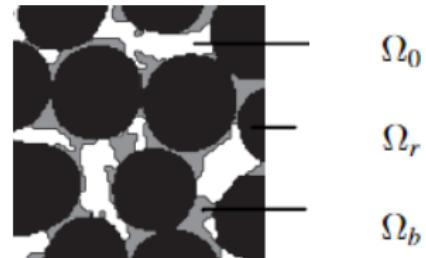


[Lewis Lab at Northeastern University.montana.edu]¹

¹ Image courtesy of the Lewis Lab at Northeastern University. Image created by Anthony D'Onofrio, William H. Fowle, Eric J. Stewart and Kim Lewis.

² Peszynska, Trykozko, Iltis, Schlueter, Wildenschild. Biofilm growth in porous media: experiments, computational modeling at the porescale, and upscaling. Advances in Water Resources, Volume 95, 2016.

Pore-Scale Domain



Ω_r : rock domain

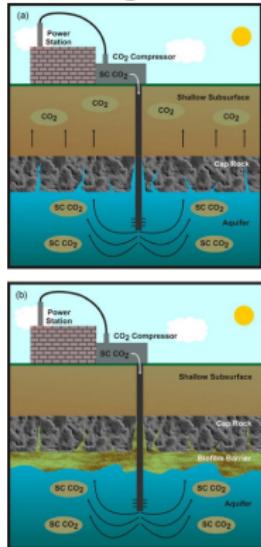
Ω_b : biofilm domain

Ω_0 : no microbes domain

Image from micro-CT from [PTISE16]²

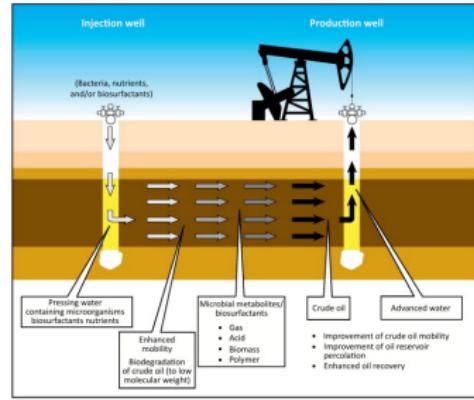
Motivation: Applications

CO₂ Sequestration



[Mitchell et al, 2009]¹

Microbial enhanced oil recovery (MEOR)



[Marchant and Banat, 2012]²

¹ Andrew C. Mitchell and Adrienne J. Phillips and Randy Hiebert and Robin Gerlach and Lee H. Spangler and Alfred B. Cunningham. Biofilm enhanced geologic sequestration of supercritical CO₂. International Journal of Greenhouse Gas Control, 2009.

² Marchant, R., and Banat, I. M. (2012). Microbial biosurfactants: challenges and opportunities for future exploitation. Trends Biotechnol. 30, 558–566. doi: 10.1016/j.tibtech.2012.07.003

Coupled Flow and Biomass-Nutrient Model

Transport model

$$\begin{aligned}\partial_t B + \nabla \cdot (\mathbf{u}B) - \nabla \cdot (d_B(B)\nabla B) + \partial I_{(-\infty, B^*]}(B) &\ni r_B(B, N); \quad x \in \Omega, t > 0, \\ \partial_t N + \nabla \cdot (\mathbf{u}N) - \nabla \cdot (d_N(B)\nabla N) &= r_N(B, N); \quad x \in \Omega, t > 0;\end{aligned}$$

B : concentration of the microbe

$$m(N) := \kappa \frac{N}{N + k_N} \quad \text{Monod expression}$$

$B^* > 0$: maximum concentration of biomass

$$\begin{aligned}r_B^{growth}(B, N) &:= \kappa_B B m(N) \\ r_N^{use}(B, N) &:= -B m(N)\end{aligned}$$

N : concentration of nutrient

\mathbf{u} : flow velocity obtained by Brinkman flow model

$$I_{(-\infty, B^*]}(B) = \begin{cases} 0 & B \leq B^* \\ +\infty & \text{otherwise,} \end{cases} \quad \partial I_{(-\infty, B^*]} = (-\infty, B^*) \times \{0\} \cup \{B^*\} \times [0, \infty), \quad \lambda \in \partial I_{(-\infty, B^*]}$$

¹ Peszynska, Trykozko, Iltis, Schlueter, Wildenschild. Biofilm growth in porous media: experiments, computational modeling at the porescale, and upscaling. Advances in Water Resources, Volume 95, 2016.[PTISW16]

² Shin, Alhammali, Bigler, Vohra, Peszynska. Coupled flow and biomass-nutrient growth at pore-scale with permeable biofilm, adaptive singularity and multiple species[J]. Mathematical Biosciences and Engineering, 2021, 18(3): 2097-2149. doi: 10.3934/mbe.2021108

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Challenges:

- Volume Constraint
- Simulations
- Solver (SSNM)
- Low regularity
- Disparate length and time scales
- Analytical solution

¹ Peszynska, Trykozko, Iltis, Schlueter, Wildenschild. Biofilm growth in porous media: experiments, computational modeling at the porescale, and upscaling. Advances in Water Resources, Volume 95, 2016.[PTISW16]

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Mixed Formulation of Diffusion-Reaction PVI:

$$B_t - \nabla \cdot (d(x) \nabla B) + \partial I_{(-\infty, B^*]}(B) \ni f(B)$$

$$\alpha(x) \mathbf{U} + \nabla \mathbf{B} = 0 \text{ in } \Omega, \quad t > 0,$$

$$\mathbf{B}_t + \nabla \cdot \mathbf{U} + \partial I_{(-\infty, B^*]}(\mathbf{B}) \ni f(\mathbf{B}) \text{ in } \Omega, \quad t > 0,$$

$$B(s, t) = g(s, t) \text{ on } \partial\Omega, \quad t > 0$$

$$B(\cdot, 0) = B_0 \text{ in } \Omega;$$

$\alpha = d^{-1}$, d is a smooth function $0 < \nu_1 \leq d(x) \leq \nu_2$ for $x \in \Omega$, f is Lipschitz continuous function.

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Mixed formulation of PVI

- $V = H(\operatorname{div}, \Omega) := \{\mathbf{v} \in (L^2(\Omega))^d; \nabla \cdot \mathbf{v} \in L^2(\Omega)\}.$
- $K = \{q \in L^2(\Omega); q \leq B^* \text{ a.e on } \Omega\}.$
- $\partial I_K(B) = \{h \in L^2(\Omega); (h, q - B) \leq 0, \forall q \in K\}, \forall B \in K.$

$$(\alpha \mathbf{U}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, \mathbf{B}) = -(g, \mathbf{v} \cdot \mathbf{n})_{\partial\Omega}, \quad \forall \mathbf{v} \in V,$$

$$(\mathbf{B}_t, q - B) + (\nabla \cdot \mathbf{U}, q - B) \geq (f(\mathbf{B}), q - B), \quad \forall q \in K,$$

$$B(\cdot, 0) = B_0,$$

Existence & Uniqueness

Lemma (Showalter, 1997)¹, (Showalter, 2010)²

For any $B_0 \in L^2(\Omega)$, a Lipschitz continuous function $f : L^2(\Omega) \rightarrow L^2(\Omega)$, $T > 0$, there exists a unique solution $\mathbf{U} : [0, T] \rightarrow H(div, \Omega)$, $B : [0, T] \rightarrow L^2(\Omega)$.

Proof outline

- Let $V = H(div, \Omega)$, $M = L^2(\Omega)$. Define $\mathcal{A} : V \rightarrow V'$, $\mathcal{B} : V \rightarrow M'$, $\mathcal{C} : M \rightarrow M'$:

$$\mathcal{A}\mathbf{U} = \alpha(\cdot)\mathbf{U}, \quad \mathcal{B}\mathbf{U} = -\nabla \cdot \mathbf{U}, \quad \mathcal{C}B = \partial I_K(B).$$

- \mathcal{A} is maximal monotone, bounded and satisfies the growth condition

$$\mathcal{A}(U) + \|\mathcal{B}U\|_0^2 \longrightarrow +\infty \quad \text{if} \quad \|U\|_V \longrightarrow \infty,$$

- \mathcal{C} is maximal monotone, $I + \mathcal{C}$ is strictly monotone.
- \mathcal{B} is linear, continuous and has a closed range.
- $\ker \mathcal{B}' = \{0\}$.

¹Ralph E Showalter. Monotone operators in Banach space and nonlinear partial differential equations. American Math. Soc., 1997.

²R.E.Showalter. Nonlinear Degenerate Evolution Equations in Mixed Formulation. SIAM Journal on Mathematical Analysis. Vol. 42, No. 5, 2114-2131, 2010.

Fully Discrete Mixed Finite Element Approximation of PVI

- $\Omega_h = \Omega, h > 0, \mathcal{T}_h = \{T_i\}$ be a triangulation of Ω_h .
- Raviart-Thomas spaces: $V_h = RT_0, M_h = \mathcal{M}^0(\mathcal{T}_h), K_h = M_h \cap K$.
- $t_0 = 0 < t_1 < \dots < t_{N_T} = T, \Upsilon = \{t_0, \dots, t_{N_T}\}$.
- $\Delta t^n = t_n - t_{n-1}, \Delta t = \max_{\{n=1, \dots, N_T\}} \{\Delta t^n\}$.
- $q^n = q(t_n), \mathbf{v}^n = \mathbf{v}(t_n), \partial q^n = \frac{q^n - q^{n-1}}{\Delta t}$.
- Using the backward Euler scheme:

Find $B_h : \Upsilon \rightarrow K_h$, and $\mathbf{U}_h : \Upsilon \rightarrow V_h$ such that for $n = 0, \dots, N_T - 1$,

$$\begin{aligned} (\alpha \mathbf{U}_h^{n+1}, \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, B_h^{n+1}) &= -(g^{n+1}, \mathbf{v}_h \cdot \mathbf{n})_{\partial\Omega}, \quad \forall \mathbf{v}_h \in V_h, \\ (\partial B_h^n, q_h - B_h^{n+1}) + (\nabla \cdot \mathbf{U}_h^{n+1}, q_h - B_h^{n+1}) &\geq (f(B_h^{n+1}), q_h - B_h^{n+1}), \quad \forall q_h \in K_h, \\ \|B_h(0) - B_0\|_0 &\leq Ch. \end{aligned}$$

Theoretical Error Estimate for MFEM in Literature

- (Brezzi, Hager, Raviart, 1978)¹: **MFEM for EVE.**
- (Johnson, Thomée, 1981)² & (Kim, Milner, Park, 1996)³: **semi-discrete MFEM** for unconstrained parabolic PDE.
- (Alhammali, Peszynska, 2019)⁴: semi-discrete MFEM for the PVI with RT_0 :

$$\sup_{t>0} \|B(t) - B_h(t)\|_0^2 + \int_0^t \|\mathbf{U}(s) - \mathbf{U}_h(s)\|_0^2 ds \leq C(h^2).$$
- (Johnson, 1976)⁵: $B_{tt} \notin L^2(\Omega)$, **fully discrete FEM** for PVI (no reaction term)
- (Alhammali, Peszynska, 2019)⁴: FEM for PVI following
 (Johnson, 1976): $\|e^n\|_{L^\infty(L^2)} = O(h + (\log \Delta t^{-1})^{1/4} \Delta t^{3/4}).$
- (Arbogast, Wheeler, Zhang, 1996)⁶: **fully discrete MFEM** for unconstrained nonlinear degenerate parabolic PDE, time integration strategy.

¹ Franco Brezzi and William W. Hager and P. A. Raviart. Error estimates for the finite element solution of variational inequalities. part II. Mixed Methods. Numerische Mathematik. Vol(31), No.1, 1978.

² Johnson, Thomée. Error estimates for some mixed finite element methods for parabolic type problems. 1981.

³ M.-Y. Kim and F.A. Milner and E.-J. Park. Some observations on mixed methods for fully nonlinear parabolic problems in divergence form. Applied Mathematics Letters. Vol(9), No. 1, 1996.

⁴ A. Alhammali, Numerical Analysis of a System of Parabolic Variational Inequalities with Application to Biofilm Growth, Ph.D thesis, Oregon State University, 2019

⁵ C. Johnson. A convergence estimate for an approximation of a parabolic variational inequality. SIAM J. Numer. Anal. 13, no. 4, 599a606, 1976

⁶ Todd Arbogast and Mary F. Wheeler and Nai-Ying Zhang. A Nonlinear Mixed Finite Element Method for a Degenerate Parabolic Equation Arising in Flow in Porous Media. SIAM Journal on Numerical Analysis. Vol.33, No.4, 1996.

Fully Discrete MFEM for PVI Following Time Integration Strategy ¹:

Integrating in time from 0 to $t \in (0, T)$ of:

$$\mathbf{B}_t + \nabla \cdot \mathbf{U} + \partial I_{(-\infty, B^*]}(\mathbf{B}) \ni f(\mathbf{B}) \text{ in } \Omega, \quad t > 0,$$

$$\alpha(x)\mathbf{U} + \nabla \mathbf{B} = 0 \text{ in } \Omega, \quad t > 0,$$

$$\mathbf{B}(s, t) = g(s, t) \text{ on } \partial\Omega, \quad t > 0$$

$$\mathbf{B}(\cdot, 0) = \mathbf{B}_0 \text{ in } \Omega;$$

\Rightarrow

$$\begin{aligned} (\mathbf{B}, q - \mathbf{B}) + (\nabla \cdot \int_0^t \mathbf{U}(\tau) d\tau, q - \mathbf{B}) &\geq (\int_0^t f(\mathbf{B}(\tau)) d\tau, q - \mathbf{B}) + (\mathbf{B}_0, q - \mathbf{B}) \quad \forall q \in K, \\ (\alpha(x)\mathbf{U}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, \mathbf{B}) &= 0 \quad \forall \mathbf{v} \in V. \end{aligned}$$

MFEM in space & backward Euler in time

$$\begin{aligned} (\mathbf{B}_h^n, q_h - \mathbf{B}_h^n) + (\nabla \cdot \sum_{j=1}^n \mathbf{U}_h^j \Delta t^j, q_h - \mathbf{B}_h^n) &\geq (\sum_{j=1}^n f(\mathbf{B}_h^j) \Delta t^j, q_h - \mathbf{B}_h^n) + (B_{h,0}, q_h - \mathbf{B}_h^n) \quad \forall q_h \in K_h, \\ (\alpha(x)\mathbf{U}_h^n, \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, \mathbf{B}_h^n) &= 0 \quad \forall \mathbf{v}_h \in V_h, \\ \|B_{h,0} - B_0\|_0 &\leq Ch. \end{aligned}$$

¹ T. Arbogast, M.F. Wheeler, and N-Y. Zhang. A nonlinear mixed finite element method for a degenerate parabolic equation arising in flow in porous media. SIAM Journal on Numerical Analysis, 33(4): 1669-1687, 1996

Error Estimate

Theorem [Alhammali, Peszynska, 2022]

$$\sum_{j=1}^n \|B^j - B_h^j\|_0^2 \Delta t^j + \left\| \int_0^{t_n} \mathbf{U}(\tau) d\tau - \sum_{j=1}^n \mathbf{U}_h^j \Delta t^j \right\|_0^2 \leq C \left(h^2 + (\Delta t)^2 \right);$$

$$B \in W^{1,\infty}(J; L^\infty(\Omega)) \cap W^{1,\infty}(J; L^2(\Omega)) \cap L^\infty(J; H^2(\Omega)), U \in L^\infty(J; (H^1(\Omega))^2),$$

$$\nabla \cdot U \in L^\infty(J; H^1(\Omega)) \Rightarrow \int_0^t \mathbf{U}(\tau) d\tau \in H^1((L^2(\Omega))^d) \cap L^2(H(\Omega; \text{div})).$$

Auxiliary properties

- $\hat{\mathcal{P}}_h : (L^2(\Omega))^d \rightarrow V_h$ is the weighted $(L^2(\Omega))^d$ projection operator defined as:
 $(\alpha(x)(\hat{\mathcal{P}}_h \mathbf{v} - \mathbf{v}), \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in V_h.$
- Interpolation operator $\rho_h : (H^1(\Omega))^d \rightarrow RT_0$: $\int_e (\mathbf{v} - \rho_h \mathbf{v}) \cdot \mathbf{n} = 0 \quad \forall e \subset \partial T, \quad \forall \mathbf{v} \in (H^1(\Omega))^d.$
- The orthogonal L^2 -projection onto M_h : $\pi_h : L^2 \rightarrow M_h$: $(q - \pi_h q, \mu_h) = 0, \quad \forall \mu_h \in M_h.$
- $(\nabla \cdot \rho_h \mathbf{v}, q_h) = (\nabla \cdot \mathbf{v}, q_h) \quad \forall q_h \in M_h, \quad \forall \mathbf{v} \in V,$
- $\|\rho_h \mathbf{v} - \mathbf{v}\|_0 \leq Ch |\mathbf{v}|_1$ if $\mathbf{v} \in (H^1(\Omega))^d$,
- $\|\nabla \cdot \rho_h \mathbf{v}\|_0 \leq C \|\nabla \cdot \mathbf{v}\|_0 \quad \forall \mathbf{v} \in V,$
- $\|\nabla \cdot (\mathbf{v} - \rho_h \mathbf{v})\|_0 \leq Ch |\nabla \cdot \mathbf{v}|_1$ if $\nabla \cdot \mathbf{v} \in H^1(\Omega)$,
- $\|\pi_h q - q\|_0 \leq Ch |q|_1$ if $q \in H^1$.

Proof Outline.

- Define $\bar{\mathbf{U}}^n = \frac{1}{\Delta t^n} \int_{t_{n-1}}^{t_n} \mathbf{U}(\tau) d\tau$, $\hat{\bar{\mathbf{U}}}^n = \sum_{j=1}^n \bar{\mathbf{U}}^j \Delta t^j = \int_0^{t_n} \mathbf{U}(\tau) d\tau$
- $\sigma^n = \bar{\mathbf{U}}^n - \mathbf{U}_h^n$, $\hat{\sigma}^n = \sum_{j=1}^n \sigma^j \Delta t^j = \int_0^{t_n} \mathbf{U}(\tau) d\tau - \sum_{j=1}^n \mathbf{U}_h^j \Delta t^j$
- $e^n = B^n - B_h^n$.
- $(B^n, B^n - B_h^n) \leq (\nabla \cdot \hat{\bar{\mathbf{U}}}^n, B_h^n - B^n) + \left(\int_0^{t_n} f(B(\tau)) d\tau, B^n - B_h^n \right) + (B_0, B^n - B_h^n)$
- $-(B_h^n, \pi_h B^n - B_h^n) \leq (\nabla \cdot \sum_{j=1}^n \mathbf{U}_h^j \Delta t^j, \pi_h B^n - B_h^n) + \left(\sum_{j=1}^n f(B_h^j) \Delta t^j, B_h^n - \pi_h B^n \right) + (B_{h,0}, B_h^n - \pi_h B^n)$.
- Take $\mathbf{v}_h = \hat{\mathcal{P}}_h \hat{\sigma}^n = \rho_h \hat{\sigma}^n - \rho_h \hat{\bar{\mathbf{U}}}^n + \hat{\mathcal{P}}_h \hat{\bar{\mathbf{U}}}^n$ in

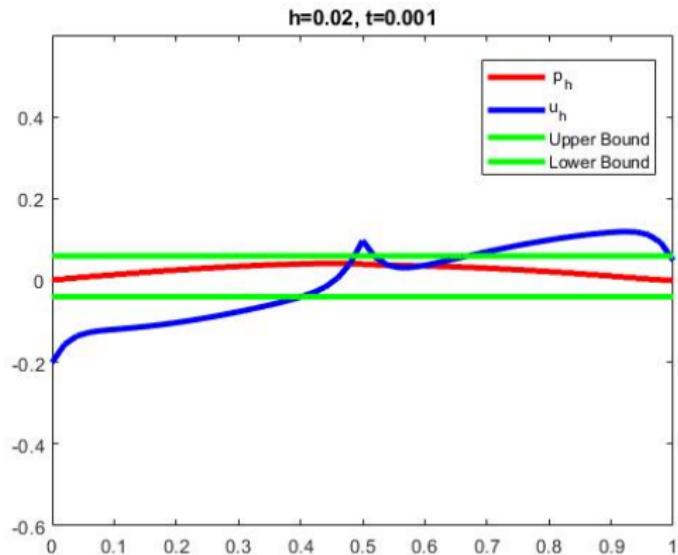
$$(\alpha(x)(\mathbf{U}^n - \mathbf{U}_h^n), \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, B - B_h^n) = 0, \quad \forall \mathbf{v}_h \in V_h.$$

- Combine & replace n by j and multiply by Δt^j and take the sum from 1 through n :

$$\sum_{j=1}^n \|e^j\|_0^2 \Delta t^j + \frac{1}{2} (\alpha(x) \hat{\mathcal{P}}_h \hat{\sigma}^n, \hat{\mathcal{P}}_h \hat{\sigma}^n) \leq \sum_{k=1}^6 T_k,$$

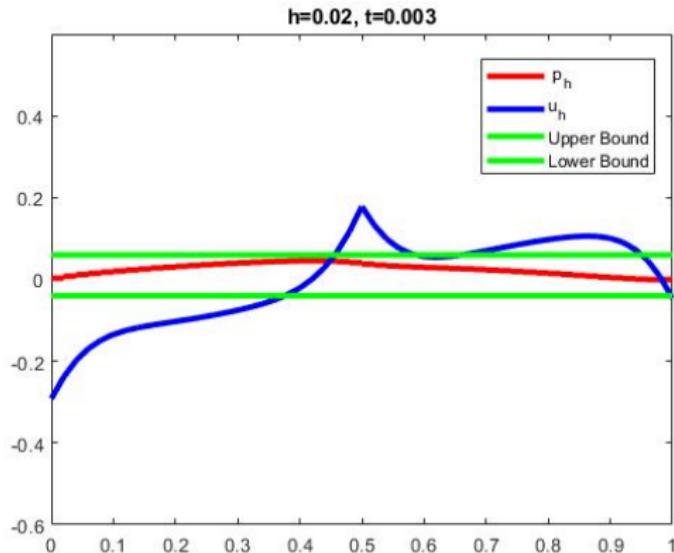
- Use auxiliary properties and Gronwall's Lemma.

1D Example: $p_t - \nabla \cdot (d(x)\nabla p) + \partial I_{(-0.04, 0.06]}(p) \ni f(p), \text{ in } \Omega$



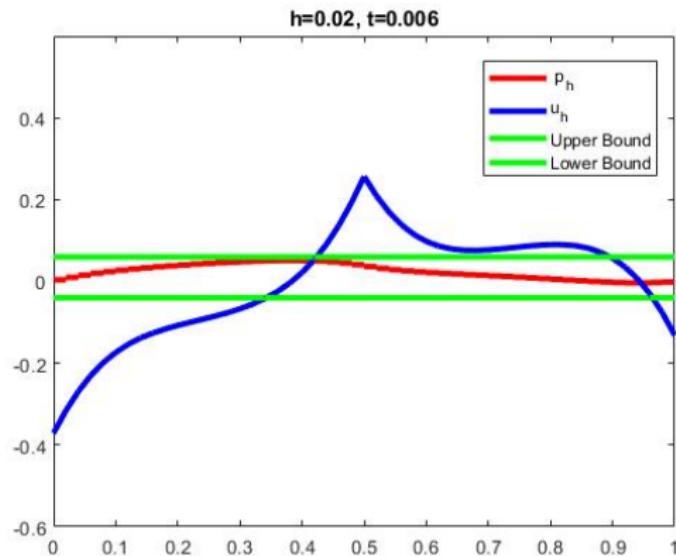
$$d(x) = 1, f(p) = \pi^2 \sin(\pi x)p + 3H(x - 0.5) - 3H(0.5 - x), p_0 = 0.04\sin(\pi x)$$

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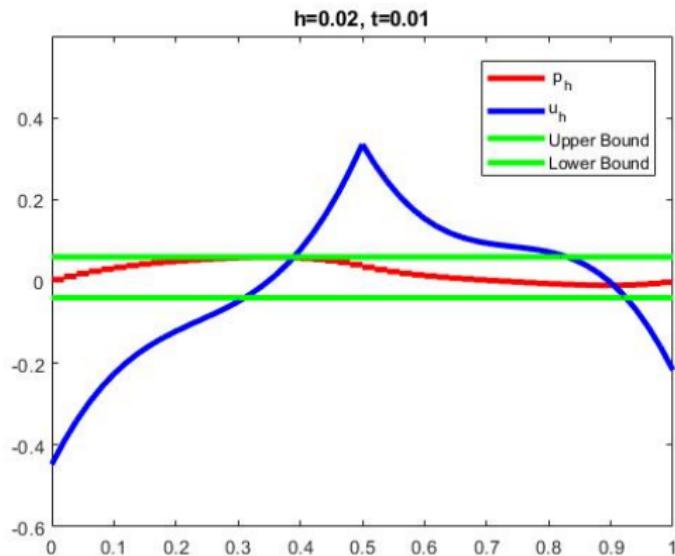
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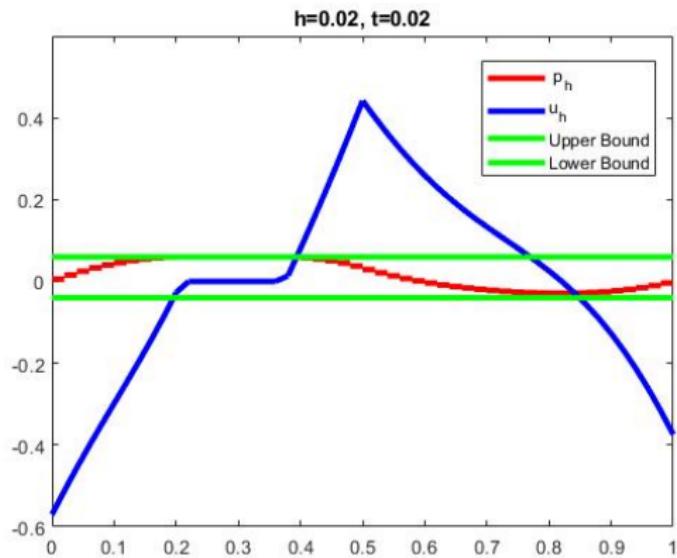
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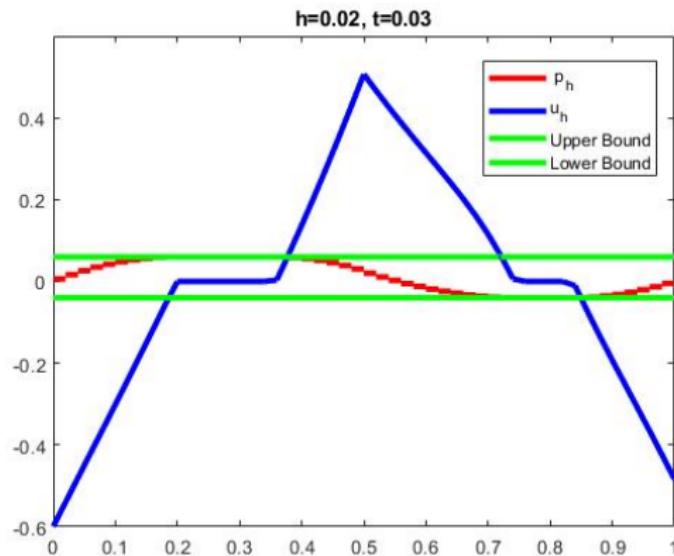
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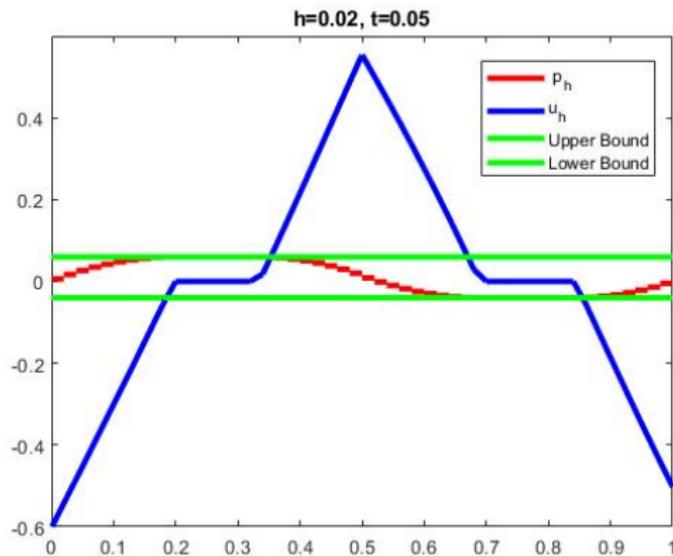
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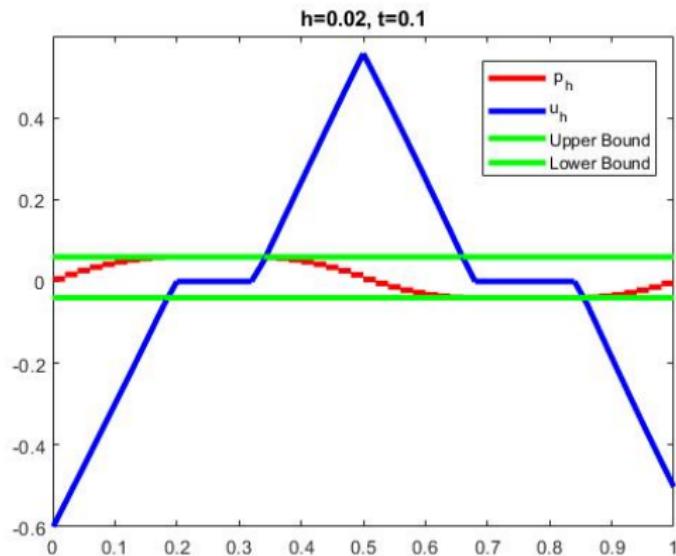
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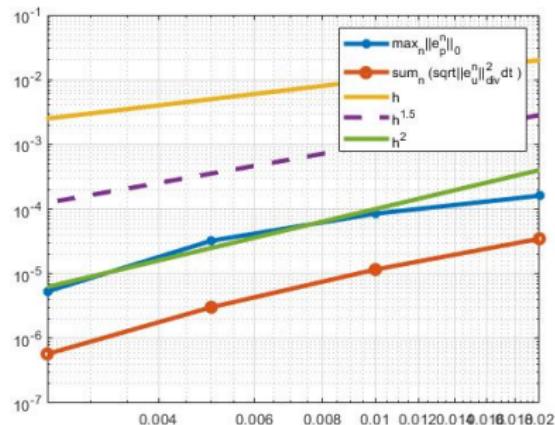
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1D Example: $p_t - \nabla \cdot (d(x)\nabla p) + \partial I_{(-0.04, 0.06]}(p) \geq f(p), \text{ in } \Omega$

h	dt	Err_p	Err_u	Order-p	Order-u
0.02	0.000128	0.00016058	3.4676e-05	-	-
0.01	6.4e-05	8.4848e-05	1.1614e-05	0.92034	1.578
0.005	3.2e-05	3.2331e-05	3.0355e-06	1.392	1.9359
0.0025	3.2e-05	5.3221e-06	5.7526e-07	2.6029	2.3997

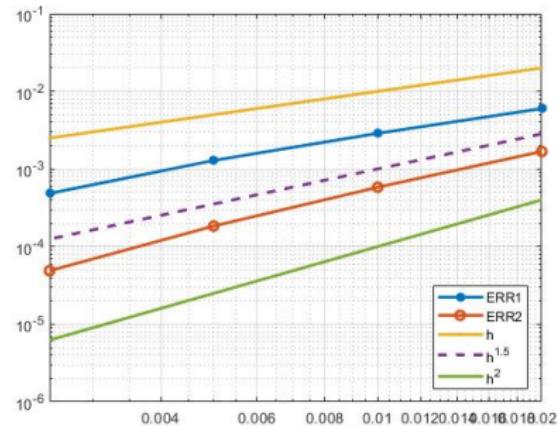
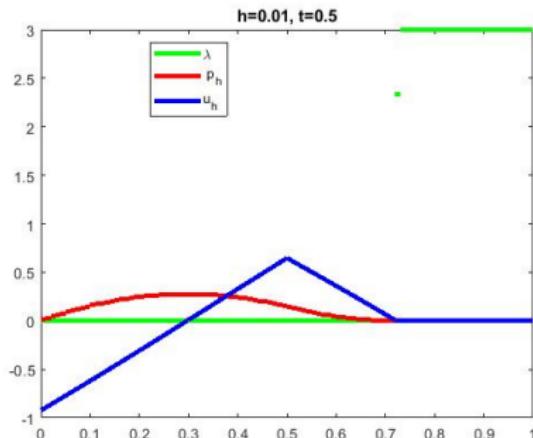


$$Err_p = \max_{\{N_{0.03}, N_{0.05}\}} \|p_{fine}^n - p_h^n\|_0, Err_u = \sqrt{\sum_{\{N_{0.03}, N_{0.05}\}} \|u_{fine}^n - u_h^n\| \Delta t},$$

$(p_{fine}, u_{fine}) \text{ at } dt = 10^{-5}, h = 0.001.$

1D Example: $p_t - \nabla \cdot (d(x)\nabla p) + \partial I_{(-\infty, 0]}(p) \ni f(p), \text{ in } \Omega$

$$d(x) = 0.5, f(x, t, p) = 3H(x - 0.5) - 3H(0.5 - x), p_0 = \sin(\pi x)$$



h	dt	ERR_p	ERR_u	Order ERR_p	Order ERR_u
0.02	0.01	0.005992	0.0016802	-	-
0.01	0.005	0.0028775	0.00058142	1.0582	1.531
0.005	0.0025	0.0012866	0.00018382	1.1613	1.6613
0.0025	0.00125	0.00048411	4.8911e-05	1.4102	1.9101

(p_{fine}, u_{fine}) at $dt = 5 \times 10^{-4}$, $h = 0.001$, $n \in \{N_{0.2}, N_{0.5}\}$.

Fully Discrete MFEM for Advection- Diffusion-Reaction

$$\text{PVI: } B_t + \nabla \cdot (\mathbf{u}B) - \nabla \cdot (d(x)\nabla B) + \partial I_{(-\infty, B^*]}(B) \ni f(B)$$

Take: $\mathbf{U} = \mathbf{u}B - d(x)\nabla B$

\Rightarrow

$$\begin{aligned} (\mathbf{B}, q - \mathbf{B}) + (\nabla \cdot \int_0^t \mathbf{U}(\tau) d\tau, q - \mathbf{B}) &\geq (\int_0^t f(\mathbf{B}(\tau)) d\tau, q - \mathbf{B}) + (\mathbf{B}_0, q - \mathbf{B}) \quad \forall q \in K, \\ (\alpha(x)\mathbf{U}, \mathbf{v}) - (\alpha(x)\mathbf{u}\mathbf{B}, \mathbf{v}) + (\nabla \cdot \mathbf{v}, \mathbf{B}) &= 0 \quad \forall \mathbf{v} \in V. \end{aligned}$$

\Rightarrow

$$\begin{aligned} (\mathbf{B}_h^n, q_h - \mathbf{B}_h^n) + (\nabla \cdot \sum_{j=1}^n \mathbf{U}_h^j \Delta t^j, q_h - \mathbf{B}_h^n) &\geq (\sum_{j=1}^n f(\mathbf{B}_h^j) \Delta t^j, q_h - \mathbf{B}_h^n) + (B_{h,0}, q_h - \mathbf{B}_h^n) \quad \forall q_h \in K_h, \\ (\alpha \mathbf{U}_h^n, \mathbf{v}_h) - (\alpha \mathbf{u} \mathbf{B}_h^n, \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, \mathbf{B}_h^n) &= 0 \quad \forall \mathbf{v}_h \in V_h, \\ \|B_{h,0} - B_0\|_0 &\leq Ch. \end{aligned}$$

Theorem [Alhammali, Peszynska, 2022]

$$\sum_{j=1}^n \|B^j - B_h^j\|_0^2 \Delta t^j + \left\| \int_0^{t_n} \mathbf{U}(\tau) d\tau - \sum_{j=1}^n \mathbf{U}_h^j \Delta t^j \right\|_0^2 \leq C \left(h^2 + (\Delta t)^2 \right);$$

Fully Discrete MFEM for the Constrained Coupled System: $\mathcal{O}(h + \Delta t)$

$$(\mathbf{B}, q - \mathbf{B}) + (\nabla \cdot \int_0^t \mathbf{U}_B(\tau) d\tau, q - \mathbf{B}) \geq (\int_0^t r_B(\mathbf{B}(\tau), \mathbf{N}(\tau)) d\tau, q - \mathbf{B}) + (\mathbf{B}_0, q - \mathbf{B}) \quad \forall q \in K,$$

$$(\alpha_B(x) \mathbf{U}_B, \mathbf{v}) - (\alpha_B(x) \mathbf{u}_B, \mathbf{v}) + (\nabla \cdot \mathbf{v}, \mathbf{B}) = 0 \quad \forall \mathbf{v} \in V,$$

$$(\mathbf{N}, \psi) + (\nabla \cdot \int_0^t \mathbf{U}_N(\tau) d\tau, \psi) = (\int_0^t r_N(\mathbf{B}(\tau), \mathbf{N}(\tau)) d\tau, \psi) + (\mathbf{N}_0, \psi) \quad \forall \psi \in L^2(\Omega),$$

$$(\alpha_N(x) \mathbf{U}_N, \mathbf{w}) - (\alpha_N(x) \mathbf{u}_N, \mathbf{w}) + (\nabla \cdot \mathbf{w}, \mathbf{N}) = 0 \quad \forall \mathbf{w} \in V.$$

\Rightarrow

$$(\mathbf{B}_h^n, q_h - \mathbf{B}_h^n) + (\nabla \cdot \sum_{j=1}^n \mathbf{U}_{B,h}^j \Delta t^j, q_h - \mathbf{B}_h^n) \geq (\sum_{j=1}^n r_B(\mathbf{B}_h^j, \mathbf{N}_h^j) \Delta t^j, q_h - \mathbf{B}_h^n) + (B_{h,0}, q_h - \mathbf{B}_h^n) \quad \forall q_h \in K_h,$$

$$(\alpha \mathbf{U}_{B,h}^n, \mathbf{v}_h) - (\alpha_B \mathbf{u}_B^n, \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, \mathbf{B}_h^n) = 0 \quad \forall \mathbf{v}_h \in V_h,$$

$$(\mathbf{N}_h^n, \psi_h) + (\nabla \cdot \sum_{j=1}^n \mathbf{U}_{N,h}^j \Delta t^j, \psi_h) = (\sum_{j=1}^n r_N(\mathbf{B}_h^j, \mathbf{N}_h^j) \Delta t^j, \psi_h) + (N_{h,0}, \psi_h) \quad \forall \psi_h \in M_h,$$

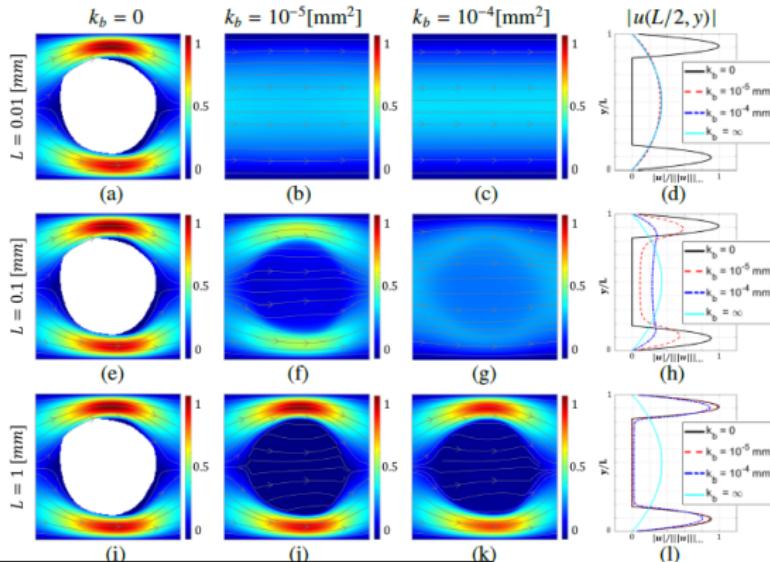
$$(\alpha_N \mathbf{U}_{N,h}^n, \mathbf{w}_h) - (\alpha_N \mathbf{u}_N^n, \mathbf{w}_h) - (\nabla \cdot \mathbf{w}_h, \mathbf{N}_h^n) = 0 \quad \forall \mathbf{w}_h \in V_h.$$

Heterogeneous Brinkman Flow Model

$$-\mu \Delta \mathbf{u} + \mu k_{bx}^{-1}(x) \mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \quad x \in \Omega_n;$$

$$k_{bx}(x) = k_b \chi_{\Omega_b}(x);$$

\mathbf{u} : flow velocity, p : pressure Ω_n : fluid domain (no rock), Ω_b : biofilm domain



[Choah Shin]¹, Flow significantly depends L and k_b ;

$\Omega = (0, L)^2$, k_b : bio-gel permeability

$$u|_{\Gamma_m} = 0, \quad u|_{\Gamma_{in}} = u_D,$$

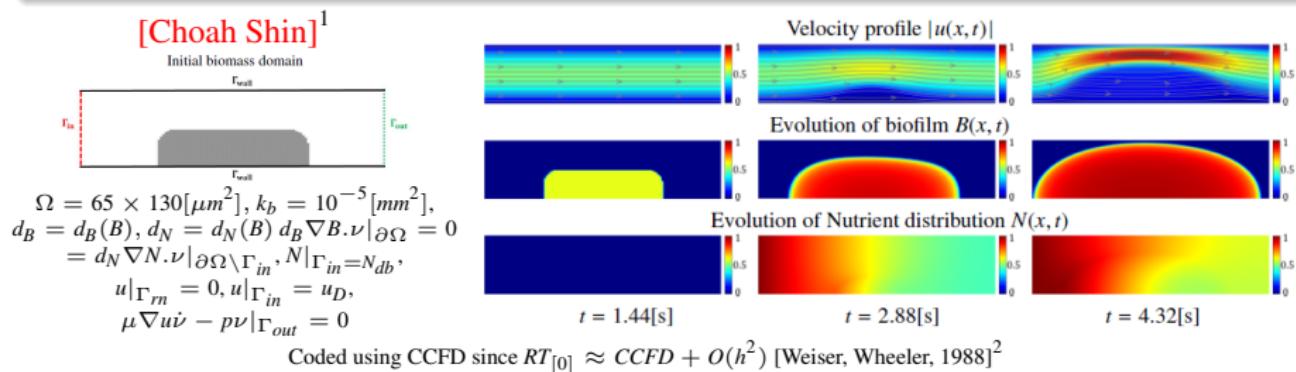
$$\mu \nabla u \cdot \nu - p \nu|_{\Gamma_{out}} = 0$$

¹ Shin, Alhammali, Bigler, Vohra, Peszynska. Coupled flow and biomass-nutrient growth at pore-scale with permeable biofilm, adaptive singularity and multiple species[J]. Mathematical Biosciences and Engineering, 2021, 18(3): 2097-2149. doi: 10.3934/mbe.2021108

Coupled Flow and Constrained Biofilm-Nutrient model

$$\begin{aligned} -\mu \Delta \mathbf{u} + \mu k_{bx}^{-1}(x) \mathbf{u} + \nabla p &= f, \quad \nabla \cdot \mathbf{u} = 0, \quad x \in \Omega_n, \\ \partial_t B + \nabla \cdot (\mathbf{u} B) - \nabla \cdot (d_B \nabla B) + \partial I_{(-\infty, B^*]}(B) &\ni r_B(B, N); \quad x \in \Omega_n, t > 0, \\ \partial_t N + \nabla \cdot (\mathbf{u} N) - \nabla \cdot (d_N \nabla N) &= r_N(B, N); \quad x \in \Omega_n, t > 0; \end{aligned}$$

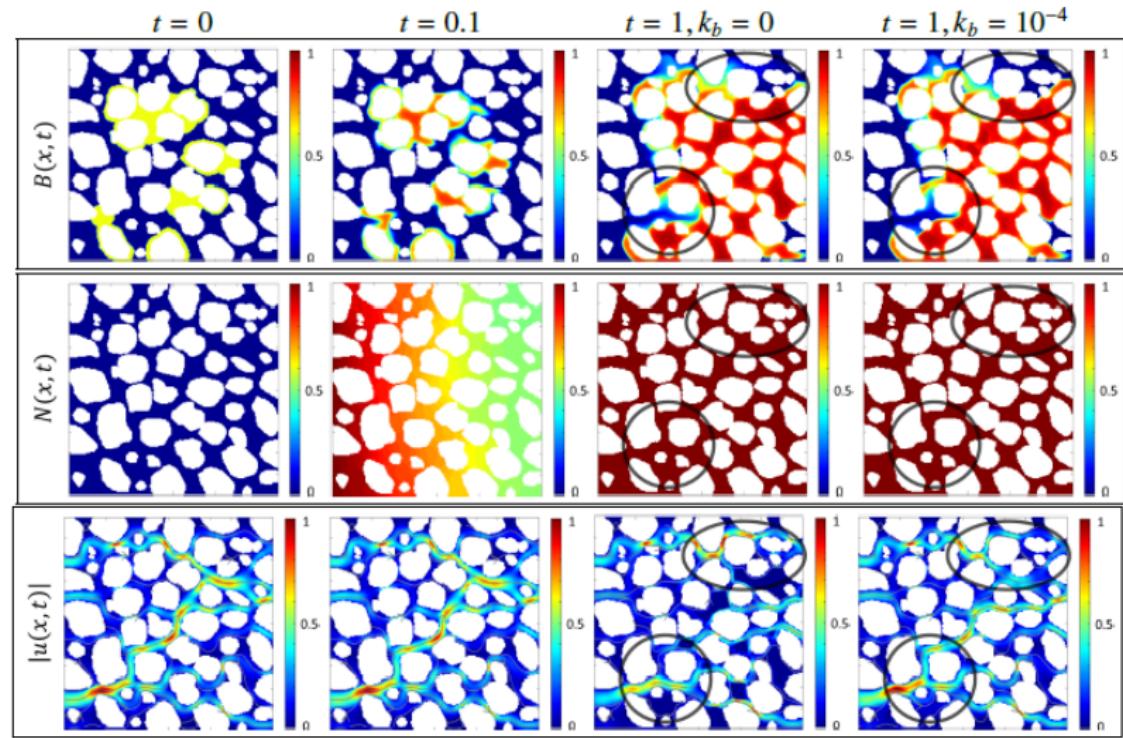
In every time step: flow \rightarrow advection \rightarrow reaction-diffusion.



¹ Shin, Alhammali, Bigler, Vohra, Peszynska. Coupled flow and biomass-nutrient growth at pore-scale with permeable biofilm, adaptive singularity and multiple species[J]. Mathematical Biosciences and Engineering, 2021, 18(3): 2097-2149. doi: 10.3934/mbe.2021108

² Alan Weiser and Mary Fanett Wheeler. On Convergence of Block-Centered Finite Differences for Elliptic Problems. SIAM Journal on Numerical Analysis. Vol. 25, No. 2, 1988.

Example: Differences in Biofilm Growth Patterns [Choah Shin]¹



¹ Shin, Alhammali, Bigler, Vohra, Peszynska. Coupled flow and biomass-nutrient growth at pore-scale with permeable biofilm, adaptive singularity and multiple species[J]. Mathematical Biosciences and Engineering, 2021, 18(3): 2097-2149. doi: 10.3934/mbe.2021108

Summary & Current Work

Summary:

- Drive an error estimate for fully discrete MFEM for the constrained coupled B–N model with advection term.
- Illustrate the effect of L and k_b on Brinkman flow and on the growth of biofilm.

Current & Future work:

Consider $d_B = d_B(B)$, $d_N = d_N(B)$ in the analysis.