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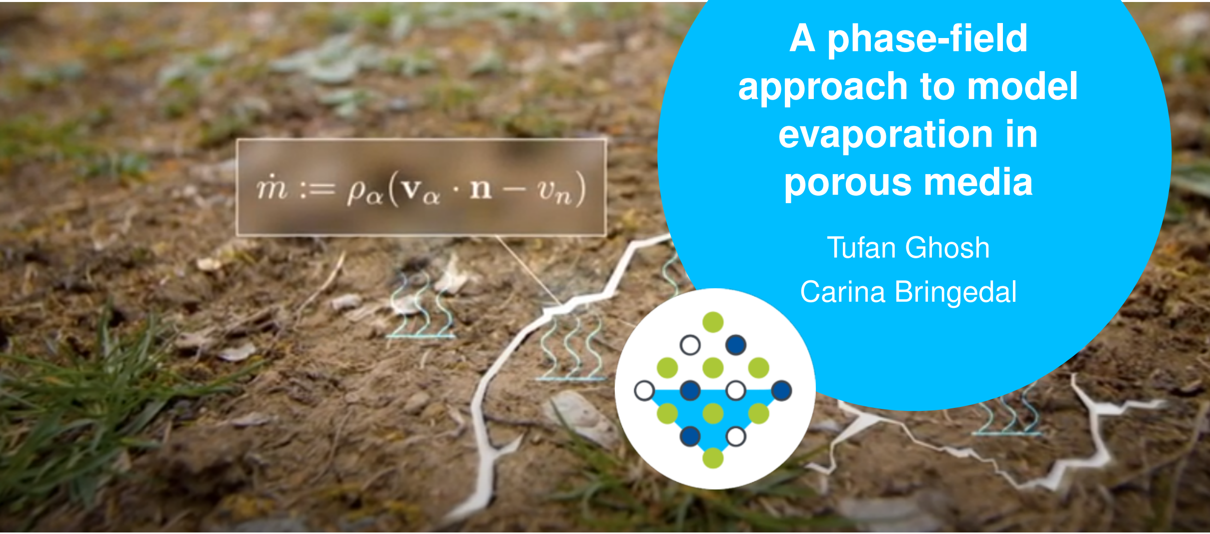
Institute for Modelling Hydraulic and Environmental Systems (IWS)




# A phase-field approach to model evaporation in porous media

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SFB 1313 InterPore

*InterPore 2022*, 1 June 2022

# Introduction

## Motivation

- Evaporation and drying in porous media occur in many environmental and industrial systems
- Moving fluid-fluid interface at the pore-scale

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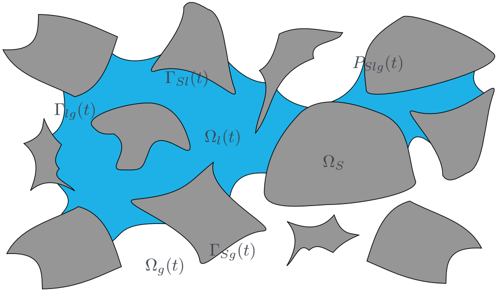


## Objectives

- Formulate a mathematical model of the relevant processes at the pore scale, including a better description of the evolving liquid-gas interface
- Derive effective models valid at the REV scale through upscaling the pore-scale processes

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- Challenge: location not known a-priori, depends on the model unknowns
- Question: how does the normal velocity of the interface $v_n$ behave?
- Information: this velocity is directly linked to the mass transfer across the interface
- Requirement: some kinematic condition coupling the normal velocity to an evaporation rate

# Sharp Interface Formulation

- Mass conservation -

$$\partial_t \rho_\alpha + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha) = 0 \qquad \text{in } \Omega_\alpha(t), \quad \alpha = l, g \tag{1}$$

$$(\rho_l \mathbf{v}_l - \rho_g \mathbf{v}_g) \cdot \mathbf{n} = v_n (\rho_l - \rho_g) \qquad \text{on } \Gamma_{lg}(t) \tag{2}$$

$$\partial_t \left(\rho_g \chi_g^v\right) + \nabla \cdot \left(\rho_g \chi_g^v \mathbf{v}_g\right) = \nabla \cdot \left(\mathrm{D}_g^v \rho_g \nabla \chi_g^v\right) \qquad \text{in } \Omega_g(t) \tag{3}$$

$$\left(\rho_l \mathbf{v}_l - \left(\rho_g \chi_g^v \mathbf{v}_g - \mathrm{D}_g^v \rho_g \nabla \chi_g^v\right)\right) \cdot \mathbf{n} = v_n \left(\rho_l - \rho_g \chi_g^v\right) \qquad \text{on } \Gamma_{lg}(t) \tag{4}$$

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- Momentum balance -

$$\partial_t \rho_\alpha + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha) = -\nabla p_\alpha + \nabla \cdot \mathcal{T}_\alpha + \rho \mathbf{g} \qquad \text{in } \Omega_\alpha(t), \ \ \alpha = l, g \tag{5}$$

$$(-(p_l - p_g)\mathbf{I} + (\mathcal{T}_l - \mathcal{T}_g)) \cdot \mathbf{n} = \dot{m}(\mathbf{v}_l - \mathbf{v}_g) - \sigma \kappa \mathbf{n} \qquad \text{on } \Gamma_{lg}(t) \tag{6}$$

$$\mathbf{v}_l \cdot \mathbf{t}_i = \mathbf{v}_g \cdot \mathbf{t}_i \qquad \text{on } \Gamma_{lg}(t) \tag{7}$$

where, $\mathcal{T}_\alpha := \mu_\alpha \left(\nabla \mathbf{v}_\alpha + \nabla \mathbf{v}_\alpha^T\right) + \xi_\alpha (\nabla \cdot \mathbf{v}_\alpha)\mathbf{I}$, $\dot{m} := \rho_\alpha (\mathbf{v}_\alpha \cdot \mathbf{n} - v_n)$, $\sigma$ - surface tension, $\kappa = \nabla_\Gamma \cdot \mathbf{n}$ - curvature

# Sharp Interface Formulation

- Energy balance -

$$\partial_t(\rho_\alpha \mathrm{u}_\alpha) + \nabla\cdot(\rho_\alpha \mathrm{h}_\alpha \mathbf{v}_\alpha) = \nabla\cdot(\mathrm{k}_\alpha \nabla \mathrm{T}_\alpha) + \mathbf{v}_\alpha\cdot\nabla p_\alpha + \mathcal{T}_\alpha : \nabla\mathbf{v}_\alpha \qquad \text{in } \Omega_\alpha(t),\ \ \alpha = l, g \tag{8}$$

$$(\mathrm{k}_g \nabla \mathrm{T}_g - \mathrm{k}_l \nabla \mathrm{T}_l)\cdot\mathbf{n} = \dot{m}(\mathrm{h}_g - \mathrm{h}_l) = \dot{m}\mathcal{L} \qquad \text{on } \Gamma_{lg}(t) \tag{9}$$

$$\mathrm{T}_l = \mathrm{T}_g = \mathrm{T}^{sat} \qquad \text{on } \Gamma_{lg}(t) \tag{10}$$

where, $\mathrm{u}_\alpha$ - internal energy per unit mass of phase $\alpha$, $\mathrm{h}_\alpha := \mathrm{u}_\alpha + p_\alpha/\rho_\alpha$ - specific enthalpy of phase $\alpha$, $\mathcal{L} := \mathrm{h}_g - \mathrm{h}_l$ - latent heat of evaporation

$$\partial_t(\rho_S C_{p,S} \mathrm{T}_S) = \nabla\cdot(\mathrm{k}_S \nabla \mathrm{T}_S) \qquad \text{in } \Omega_S \tag{11}$$

$$\mathrm{k}_S \nabla \mathrm{T}_S \cdot \mathbf{n}_S = \mathrm{k}_\alpha \nabla \mathrm{T}_\alpha \cdot \mathbf{n}_S \qquad \text{on } \Gamma_{S\alpha}(t) \tag{12}$$

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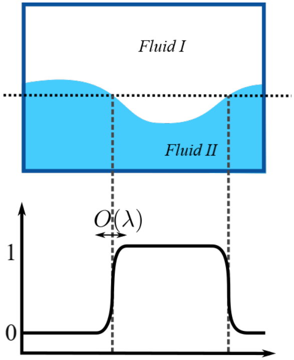
- Reaction/Evaporation rate -

$$\frac{(\rho_l + \rho_g)}{2}(v_n - \mathbf{v}_g\cdot\mathbf{n}) = -f(\chi_g^v) = -R\left\{\left(\frac{\chi_g^v}{\chi_{sat}^v}\right)^2 - 1\right\} \tag{14}$$

where, $f$ is the resulting evaporation rate, and $R$ is a reaction constant of dimension kg m$^{-2}$ s$^{-1}$

# Phase-field Formulation




- $\phi = 1$ (Fluid I), $\phi = 0$ (Fluid II)
- Approximate sharp interface by a smooth phase field $\phi$
- Moving interface replaced by a thin, diffuse layer
- All equations solved in a fixed domain


▷ Redeker, M., Rohde, C., Pop., I.S., Upscaling of a tri-phase phase-field model for precipitation in porous media, *IMA J Appl Math*, 81, 898–939, 2016.

▷ Bringedal, C., Von Wolff, L., Pop, I.S., Phase field modeling of precipitation and dissolution processes in porous media: upscaling and numerical experiments, *Multiscale Model. Simul.*, 18(2), 1076–1112, 2020.

# Diffuse Interface Model : Our Take

- Phase-field equation -

$$\rho\left\{\partial_t\phi + \nabla\cdot(\phi\mathbf{v})\right\} = \frac{\gamma}{\nu}\left\{\nabla^2\phi - \lambda^{-2}P'(\phi)\right\} - \sqrt{2}\lambda^{-1}\phi(1-\phi)f(\chi_g^v) \qquad \text{in } \Omega_F \tag{15}$$

where, $P(\phi) = \phi^2(1-\phi)^2$, $\rho = \phi\rho_l + (1-\phi)\rho_g$, $\mathbf{v}$ - velocity of the mixture, $\gamma$ and $\nu$ are parameters having dimensions kg s$^{-2}$ and m s$^{-1}$

$$\nabla\phi\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_S \tag{16}$$

- Mass conservation equation -

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad \text{in } \Omega_F \tag{17}$$

$$\partial_t\left(\phi\rho_l + (1-\phi)\rho_g\chi_g^v\right) + \nabla\cdot\left\{\left(\phi\rho_l + (1-\phi)\rho_g\chi_g^v\right)\mathbf{v}\right\} = \nabla\cdot\left(\mathrm{D}_g^v\rho_g(1-\phi)\nabla\chi_g^v\right) \qquad \text{in } \Omega_F \tag{18}$$

$$\mathrm{D}_g^v\rho_g(1-\phi)\nabla\chi_g^v\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_S \tag{19}$$

# Diffuse Interface Model : Our Take

- Momentum balance equation -

$$\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = -\nabla p + \nabla \cdot \mathcal{T} - \nabla \cdot (\lambda \sigma \nabla \phi \otimes \nabla \phi) + \rho \mathbf{g} \qquad \text{in } \Omega_F \tag{20}$$

where, $p$ - pressure of the mixture and $\mathcal{T} = \phi \mathcal{T}_l + (1 - \phi) \mathcal{T}_g$

$$\mathbf{v} = \mathbf{0} \qquad \text{on } \Gamma_S \tag{21}$$

- Energy balance equation -

$$\partial_t (\rho \mathrm{u}) + \nabla \cdot (\rho \mathrm{h} \mathbf{v}) = \nabla \cdot (\mathrm{k} \nabla \mathrm{T}) + \mathbf{v} \cdot \nabla p + \mathcal{T} : \nabla \mathbf{v} + \mathbf{v} \cdot \{\nabla \cdot (\lambda \sigma \nabla \phi \otimes \nabla \phi)\} \qquad \text{in } \Omega_F \tag{22}$$

where, $\mathrm{u}$, $\mathrm{h}$, $\mathrm{k}$ and $\mathrm{T}$ are the internal energy, specific enthalpy, heat conductivity and Temperature of the mixture, respectively

$$\mathrm{k} \nabla \mathrm{T} \cdot \mathbf{n} = \mathrm{k}_S \nabla \mathrm{T}_S \cdot \mathbf{n} \qquad \text{on } \Gamma_S \tag{23}$$

$$\mathrm{T} = \mathrm{T}_S \qquad \text{on } \Gamma_S \tag{24}$$

# Decreasing Energy

The energy associated with the above phase-field model is given by

$$E = \int_{\Omega_F} \left( \frac{1}{2}\rho \mathbf{v}^2 + \gamma\lambda^{-1} P(\phi) + \frac{1}{2}\gamma\lambda \mid \nabla\phi \mid^2 + \rho \mathrm{u} + \rho F(\rho, \phi) \right) d\mathbf{x} \tag{25}$$

Here the density energy $\rho F(\rho, \phi)$ is defined as follows

$$\partial_\phi(\rho F(\rho, \phi)) = \sqrt{2}\nu\phi(1-\phi) f(\chi_g^v) \tag{26}$$

Then one can compute

$$\frac{d}{dt}E(t) = \int_{\Omega_F} \left( \mathbf{v} \cdot \rho \mathbf{g} - \eta \nabla \cdot (\phi \mathbf{v}) - \frac{\eta^2}{\rho\lambda\nu} - \partial_\rho \left(\rho F(\rho, \phi)\right) \nabla \cdot (\rho \mathbf{v}) \right) d\mathbf{x} \tag{27}$$

where, $\eta = \gamma\lambda^{-1}P'(\phi) - \gamma\lambda\nabla^2\phi + \sqrt{2}\nu\phi(1-\phi)f(\chi_g^v)$.

$\Rightarrow$ Decreasing energy for zero velocities, or low enough divergence.

▷ Ghosh, T., Bringedal, C., A phase-field approach to model evaporation in porous media: Upscaling from pore to Darcy scale, *arXiv preprint*, 2021.

https://arxiv.org/abs/2112.13104

# Sharp Interface Limit

- Phase field/diffuse interface model can be seen as an approximation of the sharp interface model
- Introduce the dimensionless parameter $\zeta = \lambda/L$ related to thickness of the diffuse interface region
- Investigate the behavior of the solution as $\zeta \to 0$ : We recover the sharp interface formulation!

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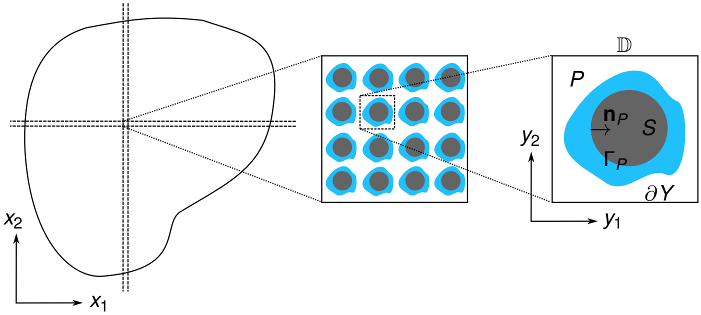
- Investigate the behavior of the solution as $\zeta \to 0$ : We recover the sharp interface formulation!
- Matched asymptotic expansion: away from the diffuse interface (outer expansion), close to it (inner expansion) and applying matching condition at the transition region
- Outer expansion: gives the governing equations in the corresponding phases Outer Expansion
- Inner expansion: provides the interface conditions between the two phases Inner Expansion

# Upscaling: Periodic Homogenization




- Scale separation: $\varepsilon = \frac{l}{L} << 1$ and $\mathbf{y} = \varepsilon^{-1}\mathbf{x}$
- Write the unknowns as a series expansion in terms of the scale separation $\varepsilon$
- Diffusion dominated regime, *i.e.* Pe = $\mathcal{O}(\varepsilon)$

# Upscaling: Periodic Homogenization

- Upscaled equations: Summary

$$\partial_t \overline{\rho}_0 + \varepsilon \nabla_{\mathbf{x}} \cdot (\overline{\rho_0 \mathbf{v}_0}) = \mathcal{O}(\varepsilon) \qquad \text{in } \mathbb{D} \qquad (28)$$

$$\partial_t \left( \overline{(\rho \chi_g^v)_0} \right) + \varepsilon \nabla_{\mathbf{x}} \cdot \left\{ \overline{(\rho \chi_g^v)_0 \mathbf{v}_0} \right\} = \nabla_{\mathbf{x}} \cdot \left( \mathcal{D} \nabla_{\mathbf{x}} \chi_{g0}^v \right) + \mathcal{O}(\varepsilon) \qquad \text{in } \mathbb{D} \qquad (29)$$

$$\overline{\mathbf{v}}_0 = -\mathcal{K} \nabla_{\mathbf{x}} p_0 - \mathcal{G} \qquad \text{in } \mathbb{D} \qquad (30)$$

$$\partial_t \left( \overline{\rho_0 \mathrm{u}_0} \right) + \partial_t \left( \rho_S C_{p,S} \mathrm{T}_{S0} \right) + \varepsilon \nabla_{\mathbf{x}} \cdot \left( \overline{\rho_0 \mathrm{h}_0 \mathbf{v}_0} \right) = \nabla_{\mathbf{x}} \cdot \left( \mathcal{A} \nabla_{\mathbf{x}} \mathrm{T}_0 \right) + \mathcal{O}(\varepsilon) \qquad \text{in } \mathbb{D} \qquad (31)$$

The phase field $\phi_0(t, \mathbf{x}, \mathbf{y})$ is updated locally in each pore by solving

$$\rho_0 \left\{ \partial_t \phi_0 + \nabla_{\mathbf{y}} \cdot (\phi_0 \mathbf{v}_0) \right\} = \frac{\gamma}{\nu} \left\{ \nabla_{\mathbf{y}}^2 \phi_0 - \lambda^{-2} P'(\phi_0) \right\} - \frac{\sqrt{2}}{\lambda} \phi_0 (1 - \phi_0) f(\chi_{g0}^v) \qquad \text{in } P$$

$$\nabla_{\mathbf{y}} \phi_0 \cdot \mathbf{n}_P = 0 \qquad \text{on } \Gamma_P \qquad (32)$$

Periodicity in $\mathbf{y}$ across $\partial Y$.

# Upscaling: Periodic Homogenization

- The effective diffusion matrix $\mathcal{D}(t, \mathbf{x})$:

$$d_{ij}(t, \mathbf{x}) = \int_P \mathrm{D}_g^v \rho_{g0}(1 - \phi_0) \left(\delta_{ij} + \partial_{y_i} \kappa^j\right) d\mathbf{y}, \qquad \text{with } i, j \in \{1, 2, \cdots, d\} \tag{33}$$

where

$$\begin{aligned} \nabla_{\mathbf{y}} \cdot \{\mathrm{D}_g^v \rho_{g0}(1 - \phi_0)(e_j + \nabla_{\mathbf{y}} \kappa^j)\} &= 0 && \text{in } P \\ \mathrm{D}_g^v \rho_{g0}(1 - \phi_0)(e_j + \nabla_{\mathbf{y}} \kappa^j) \cdot \mathbf{n}_P &= 0 && \text{on } \Gamma_P \\ \text{Periodicity in } \mathbf{y} & && \text{across } \partial Y \end{aligned} \tag{34}$$

$$\partial_t \left(\overline{(\rho \chi_g^v)_0}\right) + \varepsilon \nabla_{\mathbf{x}} \cdot \left\{\overline{(\rho \chi_g^v)_0 \mathbf{v}_0}\right\} = \nabla_{\mathbf{x}} \cdot \left(\mathcal{D} \nabla_{\mathbf{x}} \chi_{g0}^v\right) + \mathcal{O}(\varepsilon)$$

# Upscaling: Periodic Homogenization

- The effective "permeability" matrix $\mathcal{K}(t, \mathbf{x})$:

$$k_{ij}(t, \mathbf{x}) = \int_P w_i^j \, d\mathbf{y}, \qquad \text{with } i, j \in \{1, 2, \cdots, d\} \tag{35}$$

where

$$\begin{aligned}
\left(\mathbf{e}_j + \nabla_{\mathbf{y}} \Pi^j\right) + \nabla_{\mathbf{y}} \cdot \left\{\mu_0 \left(\nabla_{\mathbf{y}} \mathbf{w}^j + \nabla_{\mathbf{y}} \mathbf{w}^{j^T}\right)\right\} + \nabla_{\mathbf{y}} \cdot \left\{\xi_0 \left(\nabla_{\mathbf{y}} \cdot \mathbf{w}^j\right) \mathbf{I}\right\} &= 0 && \text{in } P \\
\nabla_{\mathbf{y}} \cdot \left(\rho_0 \mathbf{w}^j\right) &= 0 && \text{in } P \\
\mathbf{w}^j &= \mathbf{0} && \text{on } \Gamma_P \\
\text{Periodicity in } \mathbf{y} & && \text{across } \partial Y
\end{aligned} \tag{36}$$

$$\overline{\mathbf{v}}_0 = -\mathcal{K} \nabla_{\mathbf{x}} p_0 - \mathcal{G}$$

# Upscaling: Periodic Homogenization

- The effective gravity vector $\mathcal{G}(t, \mathbf{x})$:

$$g_i(t, \mathbf{x}) = \int_P w_i^0 \, d\mathbf{y}, \qquad \text{with } i \in \{1, 2, \cdots, d\} \tag{37}$$

where

$$\begin{aligned}
\nabla_{\mathbf{y}} \Pi^0 &= \nabla_{\mathbf{y}} \cdot \left\{ \mu_0 \left( \nabla_{\mathbf{y}} \mathbf{w}^0 + \nabla_{\mathbf{y}} \mathbf{w}^{0^T} \right) + \xi_0 \left( \nabla_{\mathbf{y}} \cdot \mathbf{w}^0 \right) \mathbf{I} \right\} - \nabla_{\mathbf{y}} \cdot \left( \lambda \sigma \nabla_{\mathbf{y}} \phi_0 \otimes \nabla_{\mathbf{y}} \phi_0 \right) + \rho_0 \mathbf{g} && \text{in } P \\
\nabla_{\mathbf{y}} \cdot \left( \rho_0 \mathbf{w}^0 \right) &= \partial_t \rho_0 && \text{in } P \\
\mathbf{w}^0 &= \mathbf{0} && \text{on } \Gamma_P
\end{aligned} \tag{38}$$

$$\overline{\mathbf{v}}_0 = -\mathcal{K} \nabla_{\mathbf{x}} p_0 - \mathcal{G}$$

# Upscaling: Periodic Homogenization

- The effective heat conductivity matrix $\mathcal{A}(t, \mathbf{x})$:

$$a_{ij}(t, \mathbf{x}) = \int_P \left\{ \mathrm{k}_0 \left( \delta_{ij} + \partial_{y_i} \psi^j \right) + \mathrm{k}_S \left( \delta_{ij} + \partial_{y_i} \eta^j \right) \right\} d\mathbf{y}, \qquad \text{with } i, j \in \{1, 2, \cdots, d\} \tag{39}$$

where

$$\begin{aligned}
\nabla_{\mathbf{y}} \cdot \left\{ \mathrm{k}_0 \left( e_j + \nabla_{\mathbf{y}} \psi^j \right) \right\} &= 0 && \text{in } P \\
\nabla_{\mathbf{y}} \cdot \left\{ \mathrm{k}_S \left( e_j + \nabla_{\mathbf{y}} \eta^j \right) \right\} &= 0 && \text{in } S \\
\psi^j &= \eta^j && \text{on } \Gamma_P \\
\mathrm{k}_0 \left( e_j + \nabla_{\mathbf{y}} \psi^j \right) \cdot \mathbf{n}_P + \mathrm{k}_S \left( e_j + \nabla_{\mathbf{y}} \eta^j \right) \cdot \mathbf{n}_P &= 0 && \text{on } \Gamma_P \\
\text{Periodicity in } \mathbf{y} & && \text{across } \partial Y
\end{aligned} \tag{40}$$

$$\partial_t \left( \overline{\rho_0 \mathrm{u}_0} \right) + \partial_t \left( \rho_S C_{p,S} \mathrm{T}_{S0} \right) + \varepsilon \nabla_{\mathbf{x}} \cdot \left( \overline{\rho_0 \mathrm{h}_0 \mathbf{v}_0} \right) = \nabla_{\mathbf{x}} \cdot \left( \mathcal{A} \nabla_{\mathbf{x}} \mathrm{T}_0 \right) + \mathcal{O}(\varepsilon)$$

# Summary

- Derived a new phase field model to describe evaporation on the pore scale
- Proposed model follows a decreasing free energy only for the diffusion dominated case
- Sharp interface limit of the phase field formulation recovers the sharp interface formulation (i.e., governing equations and boundary conditions)
- Derived an upscaled (REV-scale) model taking into account the pore-scale information

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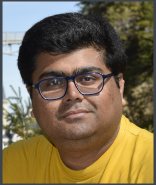
# Thank you!



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# Outer Expansion : Leading Order

- Phase field equation: $P'(\phi_0^{out}) = 0 \quad \Longrightarrow \quad \phi_0^{out} = 0,\ 1/2,\ 1.$
- Governing equations in $\Omega_0^l(t)$:

$$\partial_t \rho_{l,0}^{out} + \nabla \cdot \left( \rho_{l,0}^{out} \mathbf{v}_{l,0}^{out} \right) = 0$$

$$\partial_t \left( \rho_{l,0}^{out} \mathbf{v}_{l,0}^{out} \right) + \nabla \cdot \left( \rho_{l,0}^{out} \mathbf{v}_{l,0}^{out} \otimes \mathbf{v}_{l,0}^{out} \right) = -\nabla p_{l,0}^{out} + \nabla \cdot \mathcal{T}_{l,0}^{out} + \rho_{l,0}^{out} \mathbf{g}$$

$$\partial_t \left( \rho_{l,0}^{out} \mathrm{u}_{l,0}^{out} \right) + \nabla \cdot \left( \rho_{l,0}^{out} \mathrm{h}_{l,0}^{out} \mathbf{v}_{l,0}^{out} \right) = \nabla \cdot \left( \mathrm{k}_l \nabla \mathrm{T}_{l,0}^{out} \right) + \mathbf{v}_{l,0}^{out} \cdot \nabla p_{l,0}^{out} + \mathcal{T}_{l,0}^{out} : \nabla \mathbf{v}_{l,0}^{out}$$

- Governing equations in $\Omega_0^g(t)$:

$$\partial_t \rho_{g,0}^{out} + \nabla \cdot \left( \rho_{g,0}^{out} \mathbf{v}_{g,0}^{out} \right) = 0, \qquad \partial_t \left( \rho_{g,0}^{out} \chi_{g,0}^{v,out} \right) + \nabla \cdot \left( \rho_{g,0}^{out} \chi_{g,0}^{v,out} \mathbf{v}_{v,0}^{out} \right) = \nabla \cdot \left( \mathrm{D}_g^v \rho_{g,0}^{out} \nabla \chi_{g,0}^{v,out} \right)$$

$$\partial_t \left( \rho_{g,0}^{out} \mathbf{v}_{g,0}^{out} \right) + \nabla \cdot \left( \rho_{g,0}^{out} \mathbf{v}_{g,0}^{out} \otimes \mathbf{v}_{g,0}^{out} \right) = -\nabla p_{g,0}^{out} + \nabla \cdot \mathcal{T}_{g,0}^{out} + \rho_{g,0}^{out} \mathbf{g}$$

$$\partial_t \left( \rho_{g,0}^{out} \mathrm{u}_{g,0}^{out} \right) + \nabla \cdot \left( \rho_{g,0}^{out} \mathrm{h}_{g,0}^{out} \mathbf{v}_{g,0}^{out} \right) = \nabla \cdot \left( \mathrm{k}_g \nabla \mathrm{T}_{g,0}^{out} \right) + \mathbf{v}_{g,0}^{out} \cdot \nabla p_{g,0}^{out} + \mathcal{T}_{g,0}^{out} : \nabla \mathbf{v}_{g,0}^{out}$$

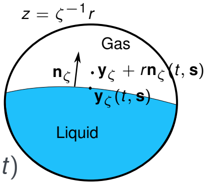
Back - Sharp-interface Limit

# Inner Expansion

- Phase field equation:

$$\mathcal{O}(\zeta^{-2}): \quad \phi_0^{in}(t,z,s) = \phi_0^{in}(z) = \frac{1}{2}\left(1 - \tanh\left(\frac{z}{\sqrt{2}L}\right)\right)$$

$$\mathcal{O}(\zeta^{-1}): \quad \frac{\left(\rho_{l,0}^{out} + \rho_{g,0}^{out}\right)}{2}\left(v_{n,0} - \mathbf{v}_{g,0}^{out} \cdot \mathbf{n}_0\right) = -\left(\frac{\gamma}{\nu}\kappa_0 + f(\chi_{g,0}^{v,out})\right) \quad \text{on } \Gamma_{lg}(t)$$




- Mass conservation equations:

$$\mathcal{O}(\zeta^{-1}): \quad v_{n,0}\left(\rho_{g,0}^{out} - \rho_{l,0}^{out}\right) = \left(\rho_{g,0}^{out}\mathbf{v}_{g,0}^{out} - \rho_{l,0}^{out}\mathbf{v}_{l,0}^{out}\right) \cdot \mathbf{n}_0 \quad \text{on } \Gamma_{lg}(t)$$

$$\mathcal{O}(\zeta^{-2}): \quad v_{n,0}\left(\rho_{g,0}^{out}\chi_{g,0}^{v,out} - \rho_{l,0}^{out}\right) = \left(\left(\rho_{g,0}^{out}\chi_{g,0}^{v,out}\mathbf{v}_{g,0}^{out} - \mathrm{D}_g^v\rho_{g,0}^{out}\nabla\chi_{g,0}^{v,out}\right) - \rho_{l,0}^{out}\mathbf{v}_{l,0}^{out}\right) \cdot \mathbf{n}_0 \quad \text{on } \Gamma_{lg}(t)$$

# Inner Expansion

- Momentum balance equation:

$$\mathcal{O}(\zeta^{-2}): \quad v_{g,0}^{out} \cdot \mathbf{t}_0 = v_{l,0}^{out} \cdot \mathbf{t}_0 \qquad \text{on } \Gamma_{lg}(t)$$

$\mathcal{O}(\zeta^{-1})$ : Provides normal velocity condition on $\Gamma_{lg}(t)$

- Energy balance equation:

$$\mathcal{O}(\zeta^{-2}): \quad \mathrm{T}_{g,0}^{out} = \mathrm{T}_{l,0}^{out} \qquad \text{on } \Gamma_{lg}(t)$$

$$\mathcal{O}(\zeta^{-1}): \quad \left(\mathrm{k}_g \nabla \mathrm{T}_{g,0}^{out} - \mathrm{k}_l \nabla \mathrm{T}_{l,0}^{out}\right) \cdot \mathbf{n}_0 = \dot{m}_0 \left(\mathrm{u}_{g,0}^{out} - \mathrm{u}_{l,0}^{out}\right) = \dot{m}_0 \mathcal{L} \qquad \text{on } \Gamma_{lg}(t)$$

▸ Back - Sharp-interface Limit