Parameter Identification in Confined Aquifers using a Predictor-Corrector Scheme of the Differential System Method

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Problem

Estimation of the transmisivity and storativity of an isotropic confined aquifer in transient conditions using water-level observations.

Model

Aquifer for which Darcy's law and the two dimensional approximation hold:

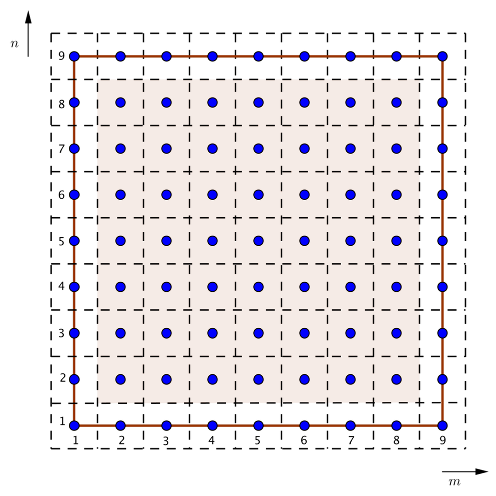
$$
\begin{aligned}
-\nabla\cdot(T\nabla h)+S\frac{\partial h}{\partial t} &= -f, \quad \text{in } \Omega,\\
h &= h_0, \quad \text{in } \partial\Omega,
\end{aligned}
$$

here,

- Ω - physical domain
- $h = h(x, y, t)$ - piezometric head
- $h_0 = h_0(x, y)$ - boundary conditions
- $T = T(x, y)$ - transmissivity
- $S = S(x, y)$ - storativity
- $f = f(x, y)$ - source term

Generation of the data

- **Study case** was presented in [1].
- **Direct problem** is solved by the Finite Volume Method.
- Then, values of h are perturbed with Gaussian noise: $m = 0, \sigma = 0.005{*}\text{mean}(h_0)$.
- **Square domain** divided into a regular lattice with spacing sides $\Delta x = \Delta y = 1000$m.



- **Transmissivity** (m^2/s)

$$T(m,n) = (18-3m+10n)\cdot 0.0005$$

- **Storativity**

$$S(m,n) = (18-3m+10n)\cdot 0.00002$$

- **Source term** (m^3/s)

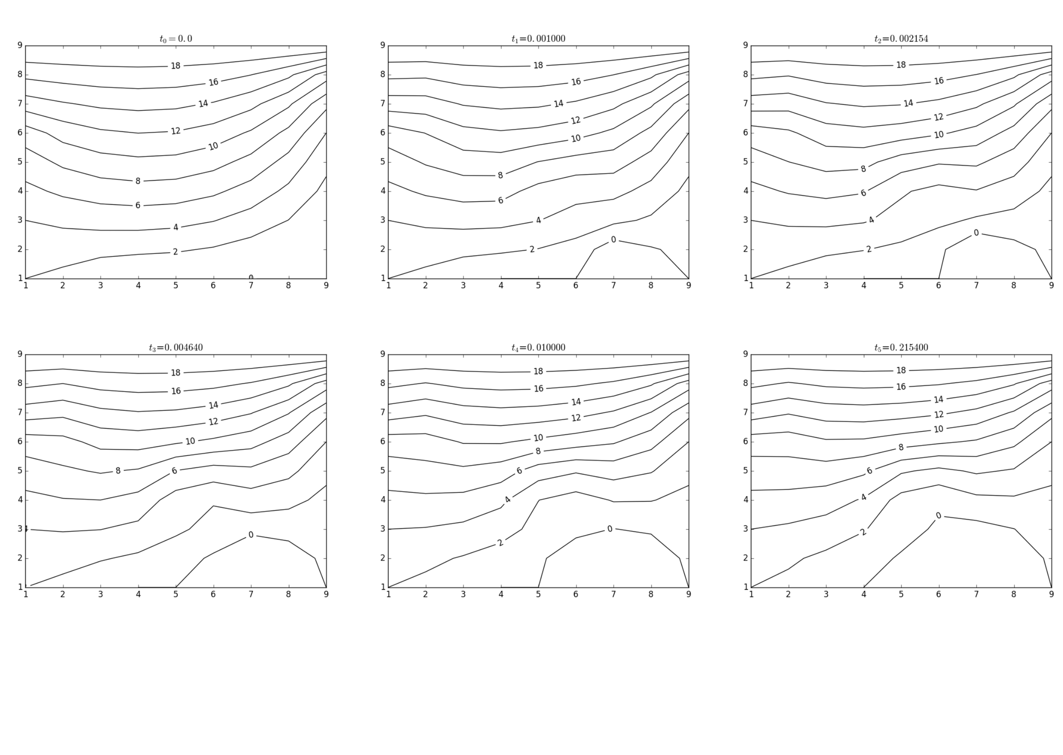
$$F(m,n) = -((4.5-m)^2+2(n-5-5)^2)\cdot 0.0005$$

Transient regime is set up by the sudden start of some wells after $t = 0$. At these wells (marked with black squares), the source term is 0.1 m^3/s.

- **Boundary conditions** of h (m).

m \ n	1	2	3	4	5	6	7	8	9
1	2.0	3.0	4.0	5.5	7.0	9.0	13.0	16.5	20.0
2	1.5	x	x	x	■	■	■	x	20.0
3	0.5	x	x	x	x	x	x	x	20.0
4	0.0	x	x	x	x	x	x	x	20.0
5	0.0	x	x	■	■	x	x	x	20.0
6	0.0	x	x	■	■	x	x	x	20.0
7	0.0	■	x	x	x	x	x	x	20.0
8	0.0	x	x	x	x	x	x	x	20.0
9	0.0	0.5	1.0	1.5	2.5	4.0	6.5	11.0	20.0

- **Piezometric head** at times $t = t_0, \ldots, t_5$ (years).



Methods

To estimate T and S at the 49 interior nodes from 81 noisy observations, we propose a method that combines two: Bayesian approach and the Differential System method.

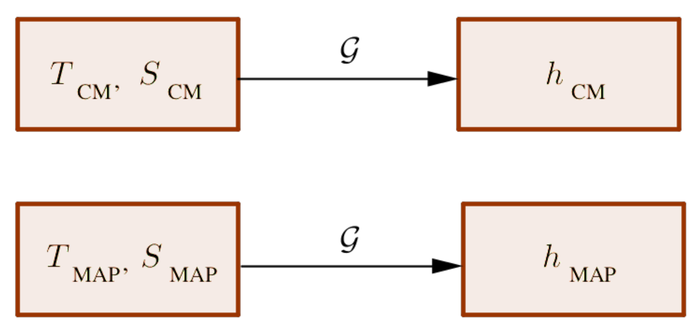
- **Bayesian approach (B)**. Parameter identification is given by the *posterior* distribution:

$$\pi(\mathbf{V}|\mathbf{p}) \propto \pi(\mathbf{p}|\mathbf{V})\pi_0(\mathbf{V}),$$

here,

- $\mathbf{V} = (\mathbf{T}, \mathbf{S})$ - parameter r. v. in $\mathbf{R}^{49\cdot 2}$
- $\mathbf{p}$ - noisy data r. v. in $\mathbf{R}^{81\cdot 6}$
- η - Gaussian noise r. v. in $\mathbf{R}^{81\cdot 6}$
- $\pi_0(\mathbf{V})$ - prior density (uniform $\mathcal{U}$ at each point)
- $\pi(\mathbf{p}|\mathbf{V}) = \pi_\eta(\mathbf{p}-\mathcal{G}(\mathbf{V}))$ - likelihood function
- $\mathcal{G}$ - forward mapping.

By sampling posterior distribution we obtain Conditional Mean (CM) and Maximum a posteriori (MAP) estimators.



- **Differential System Method (DS)**. Suppose that f, h and $\partial h/\partial t$ are known at times $t_1, \ldots, t_5$. For each $\mathbf{x} = (x, y) \in \Omega$, model equation gives

$$A\mathbf{u} = -T\mathbf{z} + \mathbf{f},$$

where

$$
\begin{aligned}
A &= \begin{pmatrix} \frac{\partial}{\partial x}h(\mathbf{x},t_1) & \frac{\partial}{\partial y}h(\mathbf{x},t_1) & \frac{\partial}{\partial t}h(\mathbf{x},t_1)\\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial x}h(\mathbf{x},t_5) & \frac{\partial}{\partial y}h(\mathbf{x},t_5) & \frac{\partial}{\partial t}h(\mathbf{x},t_5)\end{pmatrix}\\
\mathbf{u} &= \left(\frac{\partial T}{\partial x}(\mathbf{x}), \frac{\partial T}{\partial y}(\mathbf{x}), S(\mathbf{x})\right)^t\\
\mathbf{z} &= \left(\nabla^2 h(\mathbf{x},t_1), \nabla^2 h(\mathbf{x},t_2), \ldots, \nabla^2 h(\mathbf{x},t_p)\right)^t\\
\mathbf{f} &= \left(f(\mathbf{x},t_1), f(\mathbf{x},t_2), \ldots, f(\mathbf{x},t_p)\right)^t
\end{aligned}
$$

If three sets of data are independent, the system has unique l.s. solution $\mathbf{u} = (u_1, u_2, u_3)$:

$$
\begin{aligned}
u_1 &= \partial T/\partial x &= -Ta_1 + b_1\\
u_2 &= \partial T/\partial y &= -Ta_2 + b_2\\
u_3 &= S &= -Ta_3 + b_3
\end{aligned}
$$

where,

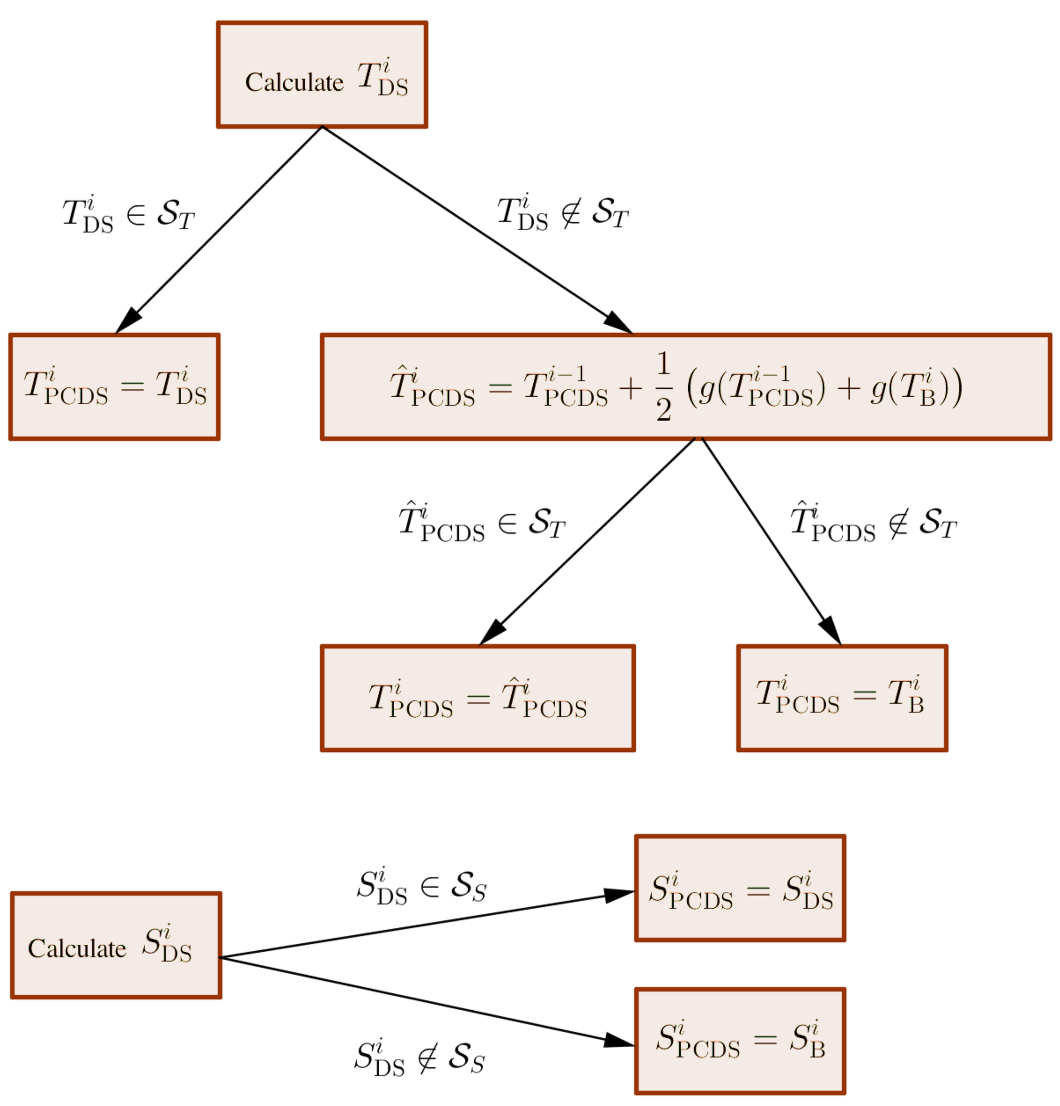
- $\mathbf{a} = (a_1, a_2, a_3)$ - l.s. solution of $A\mathbf{a} = \mathbf{z}$
- $\mathbf{b} = (b_1, b_2, b_3)$ - l.s. solution of $A\mathbf{b} = \mathbf{f}$

Identification of T at $\mathbf{x}$ is found by integrating over an adequate polygonal path joining $\mathbf{x_0}$ and the initial datum $\mathbf{x}_0$. The problem is now an ODE-problem:

$$\frac{d}{ds}T(s) = -a(s)T(s) + b(s) =: g(s),$$

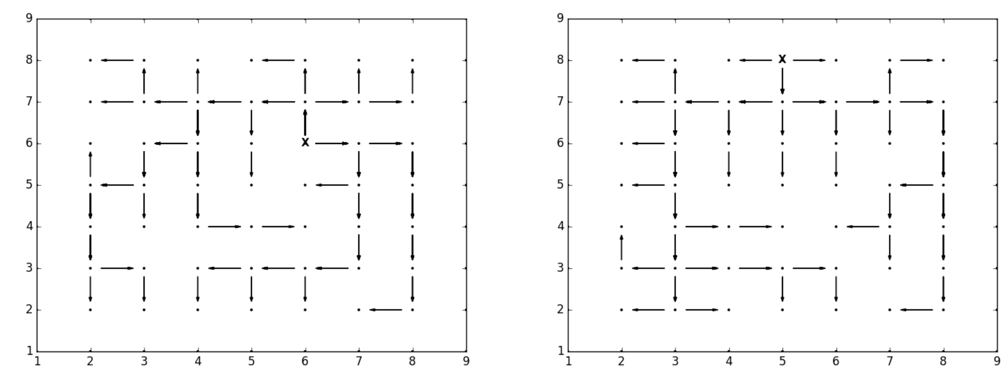
where a, b depend on a_1, a_2, b_1, b_2. Once T is estimated, the third equation gives S.

- **Predictor Corrector Scheme of the Differential System Method (PCDS)**. For our propose, we consider an appropiate support $\mathcal{S}$, and use h_{CM}, h_{MAP} instead of $\mathbf{p}$.

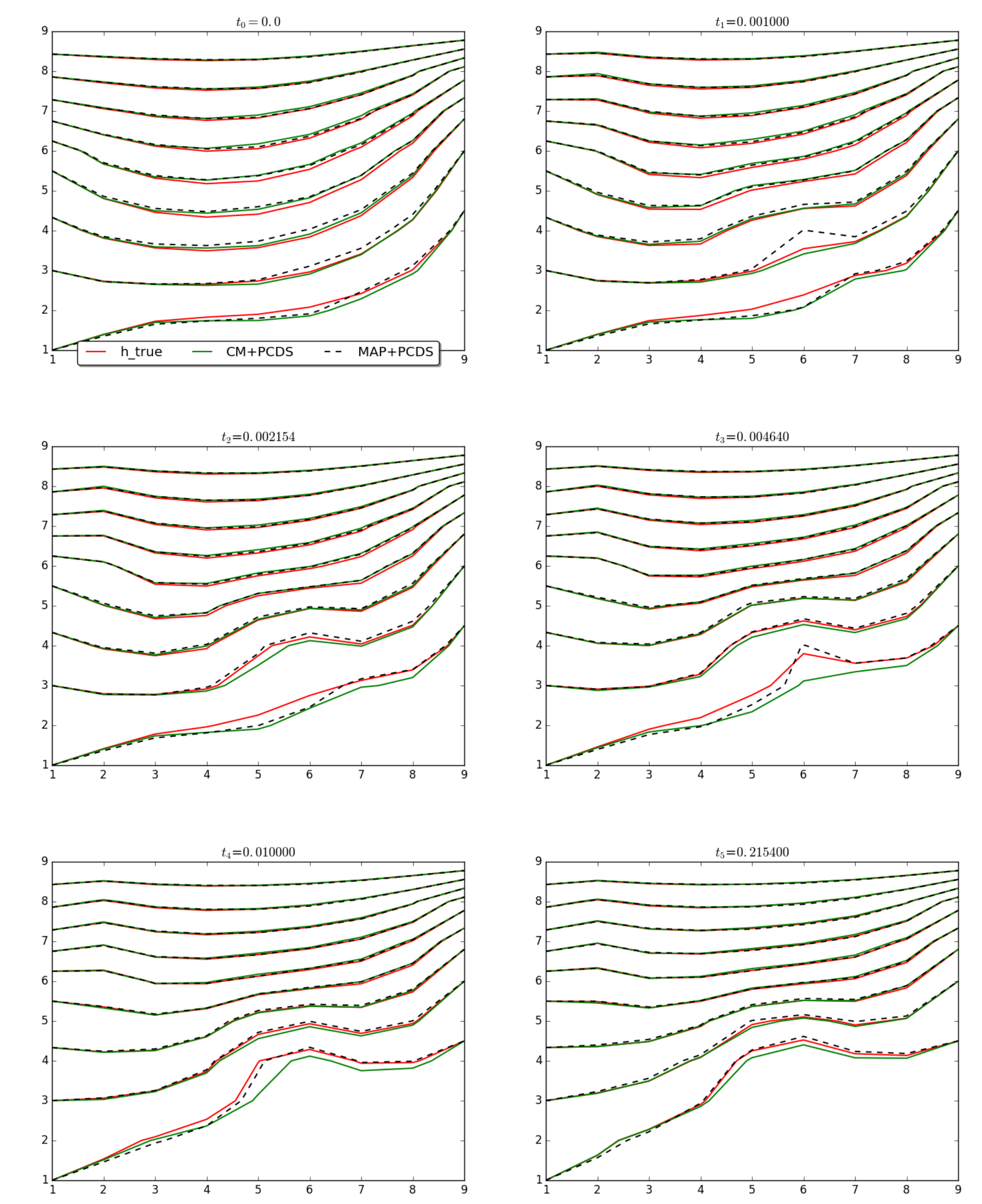


Results

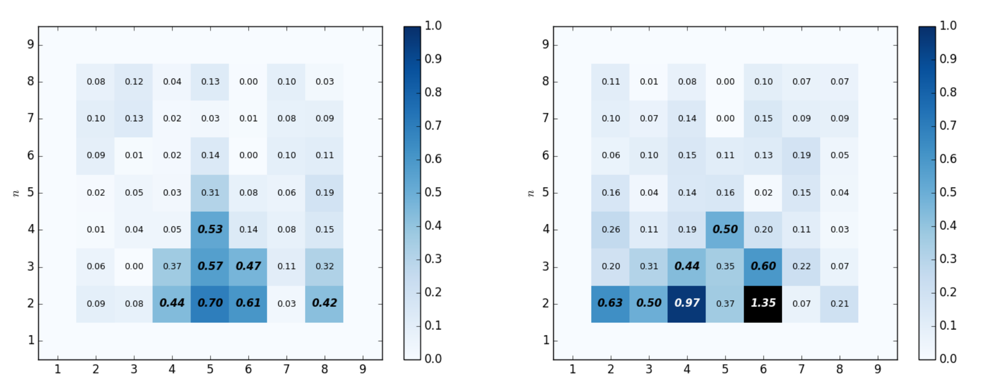
- **Integration paths**: CM at left, MAP at right.



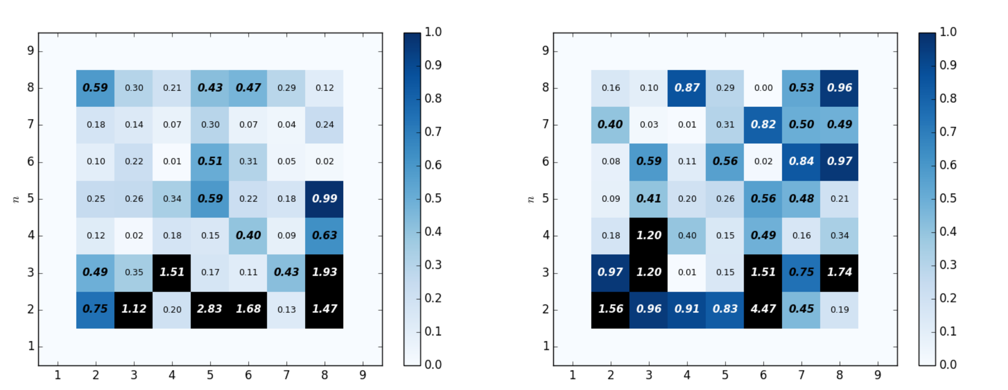
- **Piezometric head**



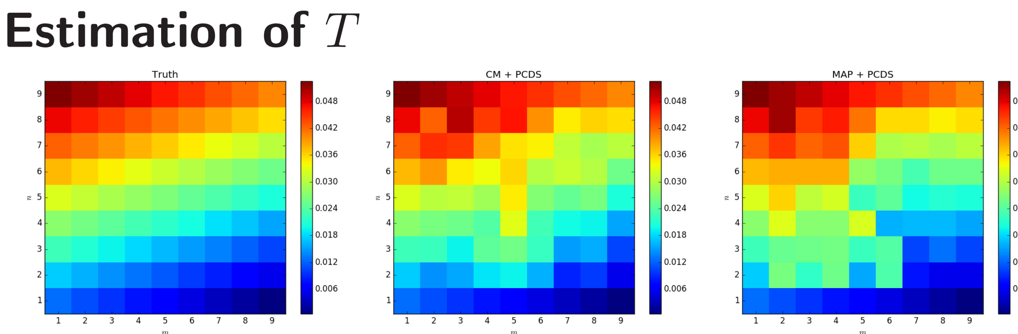
- **Relative errors for** T: CM+PCDS at left, MAP+PCDS at right.



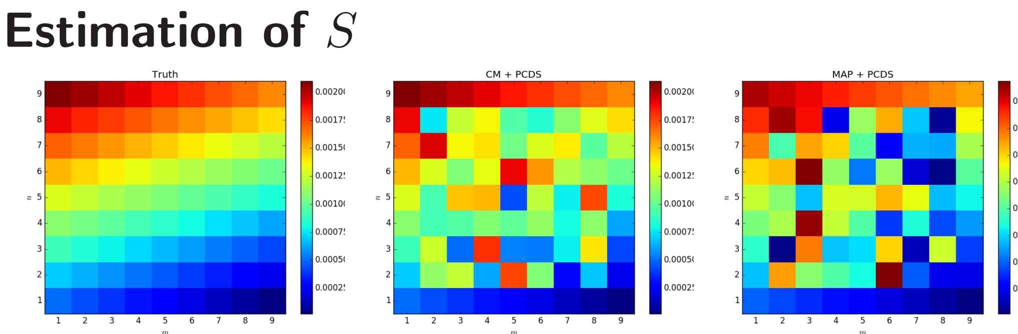
- **Relative errors for** S: CM+PCDS at left, MAP+PCDS at right.



- **Estimation of** T



- **Estimation of** S



Conclusions

Estimation of T is better by using the PCDS than using DS method or the Bayesian approach separately. Consistently with [1], the identification of S is more unstable than the identification of T.

References

[1] Vázquez, R., Giudici, M., Ponzini, G., and Parravicini, G. (1997). The differential system method for the identification of transmissivity and storativity. *Transport in Porous Media*, (26):339-371.

[2] Moreles, M. A., Vázquez, R., and Avila, F. (2004). The differential system method for parameter identification: unconfined case. *Computational Geosciences*, (8):235-253.